

MULTIPLICATION ON THE TANGENT BUNDLE

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ABSTRACT. Manifolds with a commutative and associative multiplication on the tangent bundle are called F-manifolds if a unit field exists and the multiplication satisfies a natural integrability condition. They are studied here. They are closely related to discriminants and Lagrange maps. Frobenius manifolds are F-manifolds. As an application a conjecture of Dubrovin on Frobenius manifolds and Coxeter groups is proved.

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0. INTRODUCTION

In [HM][Man](I.5) the notion of an F-manifold was introduced. It is a triple (M, \circ, e) where M is a manifold, \circ is an \mathcal{O}_M -bilinear commutative and associative multiplication on the tangent sheaf \mathcal{T}_M , e is a global unit field, and the multiplication satisfies the integrability condition

$$\mathrm{Lie}_{X \circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ) \quad (0.1)$$

for any two local vector fields X and Y . Frobenius manifolds are F-manifolds ([HM][Man](I.5)); this motivated the definition of this notion.

This paper is devoted to the local structure of complex F-manifolds. It turns out to be closely related to singularity theory. Discriminants and Lagrange maps will play a key role.

Distinguished (germs of) F-manifolds with many typical and some special properties are the base spaces of semiuniversal unfoldings of isolated hypersurface singularities and of boundary singularities (chapters 16 and 17). Here the tangent space at each parameter is canonically isomorphic to the sum of the Jacobi algebras of the singularities above this parameter. Many of the general results on F-manifolds have been known in another guise in the hypersurface singularity case and all should be compared with it.

One reason why the integrability condition (0.1) is natural is the following: Let (M, p) be the germ of an F-manifold (M, \circ, e) . The algebra $T_p M$ decomposes uniquely into a sum of (irreducible) local algebras which annihilate one another (Lemma 1.1). Now the integrability condition (0.1) ensures that this infinitesimal decomposition extends to a unique decomposition of the germ (M, p) into a product of germs of F-manifolds (Theorem 4.2).

If the multiplication at $T_p M$ is semisimple, that is, if $T_p M$ decomposes into 1-dimensional algebras, then this provides canonical coordinates u_1, \dots, u_n on (M, p) with $\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}$. In fact, at points with semisimple multiplication the integrability condition (0.1) is equivalent to the existence of such canonical coordinates. In the hypersurface case, the decomposition of the germ (M, p) for some parameter p is the unique decomposition into a product of base spaces of semiuniversal unfoldings of the singularities above p .

Another reason why (0.1) is natural is the relation to the potentiality of Frobenius manifolds. There exist F-manifolds such that not all tangent spaces are Frobenius algebras. They cannot be Frobenius manifolds. But if all tangent spaces are Frobenius algebras then the integrability condition (0.1) is related to a version of potentiality which

requires a metric on M which is multiplication invariant, but not necessarily flat. See chapter 5 for details.

This allows a viewpoint on Frobenius manifolds which was already stressed in [HM]: In the construction of some Frobenius manifolds, e.g. in the hypersurface case, the multiplication comes first and is canonical, and the most difficult property of a metric is not multiplication invariance or potentiality, but flatness.

The most important geometric object which is attributed to an n -dimensional manifold M with multiplication \circ on the tangent sheaf \mathcal{T}_M and unit field e (with or without (0.1)) is the analytic spectrum $L := \text{Specan}(\mathcal{T}_M) \subset T^*M$ (see chapter 2). The projection $\pi : L \rightarrow M$ is flat and finite of degree n . The fiber $\pi^{-1}(p) \subset L$ above $p \in M$ consists of the characters of $(T_p M, \circ)$; they correspond 1-1 to the irreducible subalgebras of $(T_p M, \circ)$ (see Lemma 1.1). The multiplication on \mathcal{T}_M can be recovered from L , because the map

$$\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L, \quad X \mapsto \alpha(\tilde{X})|_L \quad (0.2)$$

is an isomorphism of \mathcal{O}_M -algebras; here \tilde{X} is any lift of X to T^*M and α is the canonical 1-form on T^*M . The values of the function $\mathbf{a}(X)$ on $\pi^{-1}(p)$ are the eigenvalues of $X \circ : T_p M \rightarrow T_p M$.

The analytic spectrum L is a reduced variety iff the multiplication is generically semisimple. Then the manifold with multiplication (M, \circ, e) is called *massive*. Now, a third reason why the integrability condition (0.1) is natural is this: $L \subset T^*M$ is a Lagrange variety iff (M, \circ, e) is a massive F-manifold (Theorem 6.2).

The main body of this paper is devoted to study germs of massive F-manifolds at points where the multiplication is not semisimple.

We will make use of the theory of singular Lagrange varieties and their Lagrange maps, which has been worked out by Givental in [Gi2]. In fact, the notion of an irreducible germ (with respect to the decomposition above) of a massive F-manifold is equivalent to Givental's notion of a miniversal germ of a flat Lagrange map (Theorem 9.4). Via this equivalence Givental's paper contains many results on massive F-manifolds and will be extremely useful.

Locally the canonical 1-form α on T^*M can be integrated on the analytic spectrum L of a massive F-manifold (M, \circ, e) to a *generating function* $F : L \rightarrow \mathbb{C}$ which is continuous on L and holomorphic on L_{reg} . It depends on a property of L , which is weaker than normality or maximality of the complex structure of L , whether F is holomorphic on L (see chapter 7).

If F is holomorphic on L then it corresponds via (0.2) to an *Euler field* $E = \mathbf{a}^{-1}(F)$ of weight 1, that is, a vector field on M with $\text{Lie}_E(\circ) = \circ$ (Theorem 6.3).

In any case, a generating function $F : L \rightarrow \mathbb{C}$ gives rise to a Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$ (see chapters 8 and 10) and a discriminant $\mathcal{D} = \pi(F^{-1}(0)) \subset M$.

If F is holomorphic and an Euler field $E = \mathbf{a}^{-1}(F)$ exists then the discriminant \mathcal{D} is the hypersurface of points where the multiplication with E is not invertible. Then it is a free divisor with logarithmic fields $\text{Der}_M(\log \mathcal{D}) = E \circ \mathcal{T}_M$ (Theorem 13.1). This generalizes a result of K. Saito for the hypersurface case.

From the unit field e and a discriminant $\mathcal{D} \subset M$ one can reconstruct everything. One can read off the multiplication on TM in a very nice elementary way (Corollary 11.6): The e -orbit of a generic point $p \in M$ intersects \mathcal{D} in n points. One shifts the n tangent hyperplanes with the flow of e to $T_p M$. Then there exist unique vectors $e_1(p), \dots, e_n(p) \in T_p M$ such that $\sum_{i=1}^n e_i(p) = e(p)$ and $\sum_{i=1}^n \mathbb{C} \cdot e_i(p) = T_p M$ and such that the subspaces $\sum_{i \neq j} \mathbb{C} \cdot e_i(p)$, $j = 1, \dots, n$, are the shifted hyperplanes. The multiplication on $T_p M$ is given by $e_i(p) \circ e_j(p) = \delta_{ij} e_i(p)$.

In the case of hypersurface singularities and boundary singularities, the classical discriminant in the base space of a semiuniversal unfolding is such a discriminant. The critical set C in the total space of the unfolding is canonically isomorphic to the analytic spectrum L ; this isomorphism identifies the map \mathbf{a} in (0.2) with a Kodaira-Spencer map $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ and a generating function $F : L \rightarrow \mathbb{C}$ with the restriction of the unfolding function to the critical set C . This Kodaira-Spencer map \mathbf{a}_C is the source of the multiplication on \mathcal{T}_M in the hypersurface singularity case. The multiplication on \mathcal{T}_M had been defined in this way first by K. Saito.

Critical set and analytic spectrum are smooth in the hypersurface singularity case. By work of Arnold on Lagrange maps and singularities a beautiful correspondence holds (Theorem 16.6): each irreducible germ of a massive F-manifold with smooth analytic spectrum comes from an isolated hypersurface singularity, and this singularity is unique up to stable right equivalence.

By work of Nguyen huu Duc and Nguyen tien Dai the same correspondence holds for boundary singularities and irreducible germs of massive F-manifolds whose analytic spectrum is isomorphic to $(\mathbb{C}^{n-1}, 0) \times (\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0)$ with ordered components (Theorem 17.6).

The complex orbit space $M := \mathbb{C}^n / W \cong \mathbb{C}^n$ of an irreducible Coxeter group W carries an (up to some rescaling) canonical structure of a

massive F-manifold: A generating system P_1, \dots, P_n of W -invariant homogeneous polynomials induces coordinates t_1, \dots, t_n on M . Precisely one polynomial, e.g. P_1 , has highest degree. The field $\frac{\partial}{\partial t_1}$ is up to a scalar independent of any choices. This field $e := \frac{\partial}{\partial t_1}$ as unit field and the classical discriminant $\mathcal{D} \subset M$, the image of the reflection hyperplanes, determine in the elementary way described above the structure of a massive F-manifold. This follows from [Du1][Du2](Lecture 4) as well as from [Gi2](Theorem 14).

Dubrovin established the structure of a Frobenius manifold on the complex orbit space $M = \mathbb{C}^n/W$, with this multiplication, with K. Saito's flat metric on M , and with a canonical Euler field with positive weights (see Theorem 19.1). At the same place he conjectured that these Frobenius manifolds and products of them are (up to some well understood rescalings) the only massive Frobenius manifolds with an Euler field with positive weights. We will prove this conjecture (Theorem 19.3).

Crucial for the proof is Givental's result [Gi2](Theorem 14). It characterizes the germs $(M, 0)$ of F-manifolds of irreducible Coxeter groups by geometric properties (see Theorem 18.4). We obtain from it the following intermediate result (Theorem 18.3): An irreducible germ (M, p) of a *simple* F-manifold such that $T_p M$ is a Frobenius algebra is isomorphic to the germ at 0 of the F-manifold of an irreducible Coxeter group.

A massive F-manifold (M, \circ, e) is called *simple* if the germs (M, p) , $p \in M$, of F-manifolds are contained in finitely many isomorphism classes. Theorem 18.3 complements in a nice way the relation of irreducible Coxeter groups to the simple hypersurface singularities A_n, D_n, E_n and the simple boundary singularities B_n, C_n, F_4 .

In dimension 1 and 2, up to isomorphism all the irreducible germs of massive F-manifolds come from the irreducible Coxeter groups A_1 and $I_2(m)$ ($m \geq 3$) with $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(5) = H_2$, $I_2(6) = G_2$. But already in dimension 3 the classification is vast (see chapter 20).

This paper is organized as follows. In chapters 1 – 5 classical facts on finite dimensional algebras and the analytic spectrum are presented, as well as the definition of F-manifolds, the local decomposition and the relation to potentiality. In chapters 6 – 10 the Lagrange property of the analytic spectrum of a massive F-manifold and related objects as generating functions, Lyashko-Looijenga maps, and Euler fields are discussed. The chapters 11 – 15 contain a mixture of results on discriminants, 2-dimensional F-manifolds, logarithmic vector fields, modality for germs of F-manifolds and different embeddings of analytic spectra. The long final chapters 16 – 20 treat hypersurface singularities,

boundary singularities, finite Coxeter groups and 3-dimensional massive F-manifolds.

The starting point for this work was the common paper [HM] with Yu. Manin, whose interest was very encouraging and helpful. I also want to thank E. Brieskorn, my teacher in singularity theory. This paper builds on work of him, Arnold, Givental, Looijenga, Lyashko, K. Saito, O.P. Shcherbak, Teissier, and many others. The major part of this paper was written at the mathematics department of the University Paul Sabatier in Toulouse. I thank the department and especially J.-F. Mattei for their hospitality.

1. FINITE-DIMENSIONAL ALGEBRAS

In this chapter (Q, \circ, e) is a \mathbb{C} -algebra of finite dimension (≥ 1) with commutative and associative multiplication and with unit e . The next lemma gives precise information on the decomposition of Q into irreducible algebras. The statements are well known and elementary. They can be deduced directly in the given order or from more general results in commutative algebra. Algebra homomorphisms are always supposed to map the unit to the unit.

Lemma 1.1. *Let (Q, \circ, e) be as above. As the endomorphisms $x \circ : Q \rightarrow Q$, $x \in Q$, commute, there is a unique simultaneous decomposition $Q = \bigoplus_{k=1}^l Q_k$ into generalized eigenspaces Q_k (with $\dim_{\mathbb{C}} Q_k \geq 1$). Define $e_k \in Q_k$ by $e = \sum_{k=1}^l e_k$. Then*

- i) $Q_j \circ Q_k = 0$ for $j \neq k$; $e_k \neq 0$; $e_j \circ e_k = \delta_{jk} e_k$; e_k is the unit of the algebra $Q_k = e_k \circ Q$.
- ii) The function $\lambda_k : Q \rightarrow \mathbb{C}$ which associates to $x \in Q$ the eigenvalue of $x \circ$ on Q_k is an algebra homomorphism; $\lambda_j \neq \lambda_k$ for $j \neq k$.
- iii) (Q_k, \circ, e_k) is an irreducible and a local algebra with maximal ideal $\mathbf{m}_k = Q_k \cap \ker(\lambda_k)$.
- iv) The subsets $\ker(\lambda_k) = \mathbf{m}_k \oplus \bigoplus_{j \neq k} Q_j$, $k = 1, \dots, l$, are the maximal ideals of the algebra Q ; the complement $Q - \bigcup_k \ker(\lambda_k)$ is the group of invertible elements of Q .
- v) $\{\lambda_1, \dots, \lambda_l\} = \text{Hom}_{\mathbb{C}\text{-alg}}(Q, \mathbb{C})$.
- vi) the localization $Q_{\ker(\lambda_k)}$ is isomorphic to Q_k .

We call this decomposition the *eigenspace decomposition* of (Q, \circ, e) . The set $L := \{\lambda_1, \dots, \lambda_l\} \subset Q^*$ carries a natural complex structure \mathcal{O}_L such that $\mathcal{O}_L(L) \cong Q$ and $\mathcal{O}_{L, \lambda_k} \cong Q_k$. More details on this will be given in chapter 2.

The algebra (or its multiplication) is called *semisimple* if Q decomposes into 1-dimensional subspaces, $Q \cong \bigoplus_{k=1}^{\dim Q} Q_k = \bigoplus_{k=1}^{\dim Q} \mathbb{C} \cdot e_k$.

An irreducible algebra $Q = \mathbb{C} \cdot e \oplus \mathfrak{m}$ with maximal ideal \mathfrak{m} is a Gorenstein ring if the socle $\text{Ann}_Q(\mathfrak{m})$ has dimension 1.

An algebra $Q = \bigoplus_{k=1}^l Q_k$ is a Frobenius algebra if each irreducible subalgebra is a Gorenstein ring.

The next (also wellknown) lemma gives equivalent conditions and additional information. We remark that this classical definition of a Frobenius algebra is slightly weaker than Dubrovin's: he calls an algebra (Q, \circ, e) together with a *fixed* bilinear form g as in Lemma 1.2 a) iii) a Frobenius algebra.

Lemma 1.2. *a) The following conditions are equivalent.*

- i) (Q, \circ, e) is a Frobenius algebra.
- ii) $\text{Hom}(Q, \mathbb{C}) \cong Q$ as a Q -module.
- iii) There exists a bilinear form $g : Q \times Q \rightarrow \mathbb{C}$ which is symmetric, nondegenerate and multiplication invariant, i.e. $g(a \circ b, c) = g(a, b \circ c)$.

b) Let $Q = \bigoplus_{k=1}^l Q_k$ be a Frobenius algebra and $Q_k = \mathbb{C} \cdot e_k \oplus \mathfrak{m}_k$. The generators of $\text{Hom}(Q, \mathbb{C})$ as a Q -module are the linear forms $f : Q \rightarrow \mathbb{C}$ with $f(\text{Ann}_{Q_k}(\mathfrak{m}_k)) = \mathbb{C}$ for all k .

One obtains a 1-1 correspondence between these linear forms and the bilinear forms g as in a) iii) by putting $g(x, y) := f(x \circ y)$.

Proof: a) Any of the conditions i), ii), iii) in a) is satisfied for Q iff it is satisfied for each irreducible subalgebra Q_k . One checks this with $Q_j \circ Q_k = 0$ for $j \neq k$. So we may suppose that Q is irreducible.

i) \iff ii) A linear form $f \in \text{Hom}(Q, \mathbb{C})$ generates $\text{Hom}(Q, \mathbb{C})$ as a Q -module iff the linear form $(x \mapsto f(y \circ x))$ is nontrivial for any $y \in Q - \{0\}$, that is, iff $f(y \circ Q) = \mathbb{C}$ for any $y \in Q - \{0\}$.

The socle $\text{Ann}_Q(\mathfrak{m})$ is the set of the common eigenvectors of all endomorphisms $x \circ : Q \rightarrow Q$, $x \in Q$. If $\dim \text{Ann}_Q(\mathfrak{m}) \geq 2$ then for any linear form f an element $y \in (\ker f \cap \text{Ann}_Q(\mathfrak{m})) - \{0\}$ satisfies $y \circ Q = \mathbb{C} \cdot y$ and $f(y \circ Q) = 0$; so f does not generate $\text{Hom}(Q, \mathbb{C})$. If $\dim \text{Ann}_Q(\mathfrak{m}) = 1$ then it is contained in any nontrivial ideal, because any such ideal contains a common eigenvector of all endomorphisms. The set $y \circ Q$ for $y \in Q - \{0\}$ is an ideal. So, then a linear form f with $f(\text{Ann}_Q(\mathfrak{m})) = \mathbb{C}$ generates $\text{Hom}(Q, \mathbb{C})$ as a Q -module.

i) \Rightarrow iii) Choose any linear form f with $f(\text{Ann}_Q(\mathfrak{m})) = \mathbb{C}$ and define g by $g(x, y) := f(x \circ y)$. It rests to show that g is nondegenerate. But for any $x \in Q - \{0\}$ there exists a $y \in Q$ with $\mathbb{C} \cdot x \circ y = \text{Ann}_Q(\mathfrak{m})$, because $\text{Ann}_Q(\mathfrak{m})$ is contained in the ideal $x \circ Q$.

iii) \Rightarrow i) $g(\mathbf{m}, \text{Ann}_Q(\mathbf{m})) = g(e, \mathbf{m} \circ \text{Ann}_Q(\mathbf{m})) = g(e, 0) = 0$ implies $\dim \text{Ann}_Q(\mathbf{m}) = 1$.

b) Starting with a bilinear form g , the corresponding linear form f is given by $f(x) = g(x, e)$. The rest is clear from the preceding discussion. \square

The semisimple algebra $Q \cong \bigoplus_{k=1}^{\dim Q} \mathbb{C} \cdot e_k$ is a Frobenius algebra.

A classical result is that the complete intersections $\mathcal{O}_{\mathbb{C}^m, 0}/(f_1, \dots, f_m)$ are Gorenstein. But these are not all, e.g. $\mathbb{C}\{x, y, z\}/(x^2, y^2, xz, yz, xy - z^2)$ is Gorenstein, but not a complete intersection.

Finally, in the next chapter vector bundles with multiplication will be considered. Condition ii) of Lemma 1.2 a) shows that there the points whose fibers are Frobenius algebras form an open set in the base.

2. VECTOR BUNDLES WITH MULTIPLICATION

Now we consider a holomorphic vector bundle $Q \rightarrow M$ on a complex manifold M with multiplication on the fibers: The sheaf \mathcal{Q} of holomorphic sections of the bundle $Q \rightarrow M$ is equipped with an \mathcal{O}_M -bilinear commutative and associative multiplication \circ and with a global unit section e .

The set $\bigcup_{p \in M} \text{Hom}_{\mathbb{C}\text{-alg}}(Q(p), \mathbb{C})$ of algebra homomorphisms from the single fibers $Q(p)$ to \mathbb{C} (which map the unit to $1 \in \mathbb{C}$) is a subset of the dual bundle Q^* and has a natural complex structure. It is the analytic spectrum $\text{Specan}(\mathcal{Q})$. We sketch the definition [Ho](ch. 3):

The \mathcal{O}_m -sheaf $\text{Sym}_{\mathcal{O}_m} \mathcal{Q}$ can be identified with the \mathcal{O}_m -sheaf of holomorphic functions on Q^* which are polynomial in the fibers. The canonical \mathcal{O}_m -algebra homomorphism $\text{Sym}_{\mathcal{O}_m} \mathcal{Q} \rightarrow \mathcal{Q}$ which maps the multiplication in $\text{Sym}_{\mathcal{O}_m} \mathcal{Q}$ to the multiplication \circ in \mathcal{Q} is surjective. The kernel generates an ideal \mathcal{I} in \mathcal{O}_{Q^*} .

One can describe the ideal locally explicitly: suppose $U \subset M$ is open and $\delta_1, \dots, \delta_n \in \mathcal{Q}(U)$ is a base of sections of the restriction $Q|_U \rightarrow U$ with $\delta_1 = e$ and $\delta_i \circ \delta_j = \sum_k a_{ij}^k \delta_k$; denote by y_1, \dots, y_n the fiberwise linear functions on $Q^*|_U$ which are induced by $\delta_1, \dots, \delta_n$; then the ideal \mathcal{I} is generated in $Q^*|_U$ by

$$y_1 - 1 \quad \text{and} \quad y_i y_j - \sum_k a_{ij}^k y_k. \quad (2.1)$$

The support of $\mathcal{O}_{Q^*}/\mathcal{I}$ with the restriction of $\mathcal{O}_{Q^*}/\mathcal{I}$ as structure sheaf is $\text{Specan}(\mathcal{Q}) \subset Q^*$. We denote the natural projections by $\pi_{Q^*} : Q^* \rightarrow M$ and $\pi : \text{Specan}(\mathcal{Q}) \rightarrow M$. A part of the following theorem is already

clear from the discussion. A complete proof and thorough discussion can be found in [Ho](ch. 3).

Theorem 2.1. *The set $\bigcup_{p \in M} \text{Hom}_{\mathbb{C}\text{-alg}}(Q(p), \mathbb{C})$ is the support of the analytic spectrum $\text{Specan}(\mathcal{Q}) =: L$. The composition of maps*

$$\mathbf{a} : \mathcal{Q} \hookrightarrow (\pi_{Q^*})_* \mathcal{O}_{Q^*} \rightarrow \pi_* \mathcal{O}_L \quad (2.2)$$

is an isomorphism of \mathcal{O}_M -algebras and of free \mathcal{O}_M -modules of rank n , here n is the fiber dimension of $Q \rightarrow M$. The projection $\pi : L \rightarrow M$ is finite and flat of degree n .

Consider a point $p \in M$ with eigenspace decomposition $Q(p) = \bigoplus_{k=1}^{l(p)} Q_k(p)$ and $L \cap \pi^{-1}(p) = \{\lambda_1, \dots, \lambda_{l(p)}\}$. The restriction of the isomorphism

$$\mathcal{Q}_p \xrightarrow{\mathbf{a}} (\pi_* \mathcal{O}_L)_p \cong \bigoplus_{k=1}^{l(p)} \mathcal{O}_{L, \lambda_k} \quad (2.3)$$

to the fiber over p yields isomorphisms

$$Q_k(p) \cong \mathcal{O}_{L, \lambda_k} \otimes_{\mathcal{O}_{M, p}} \mathbb{C}. \quad (2.4)$$

Corollary 2.2. *In a sufficiently small neighborhood U of a point $p \in M$, the eigenspace decomposition $Q(p) = \bigoplus_{k=1}^{l(p)} Q_k(p)$ of the fiber $Q(p)$ extends uniquely to a decomposition of the bundle $Q|_U \rightarrow U$ into multiplication invariant holomorphic subbundles.*

Proof of Corollary 2.2: The $\mathcal{O}_{M, p}$ -free submodules $\mathcal{O}_{L, \lambda_k}$ in the decomposition in (2.3) of $(\pi_* \mathcal{O}_L)_p$ are obviously multiplication invariant. Via the isomorphism \mathbf{a} one obtains locally a decomposition of the sheaf \mathcal{Q} of sections of $Q \rightarrow M$ into multiplication invariant free \mathcal{O}_M -submodules. \square

Of course, the induced decomposition of $Q(q)$ for a point q in the neighborhood of p may be coarser than the eigenspace decomposition of $Q(q)$.

The base is naturally stratified with respect to the numbers and dimensions of the components of the eigenspace decompositions of the fibers of $Q \rightarrow M$. To make this precise we introduce a partial ordering \succ on the set \mathcal{P} of partitions of n :

$$\mathcal{P} := \{\beta = (\beta_1, \dots, \beta_{l(\beta)}) \mid \beta_i \in \mathbb{N}, \beta_i \geq \beta_{i+1}, \sum_{i=1}^{l(\beta)} \beta_i = n\};$$

for $\beta, \gamma \in \mathcal{P}$ define

$$\beta \succ \gamma : \iff \exists \sigma : \{1, \dots, l(\gamma)\} \rightarrow \{1, \dots, l(\beta)\} \text{ s.t. } \beta_j = \sum_{i \in \sigma^{-1}(j)} \gamma_i .$$

The partition $P(p)$ of a fiber $Q(p)$ is the partition of $n = \dim Q(p)$ by the dimensions of the subspaces of the eigenspace decomposition.

Proposition 2.3. *Fix a partition $\beta \in \mathcal{P}$. The subset $\{p \in M \mid P(p) \succ \beta\}$ is empty or an analytic subset of M .*

Proof: The partition $P(f)$ of a polynomial of degree n is the partition of n by the multiplicity of the zeros of f .

Fact: *The space $\{a \in \mathbb{C}^n \mid P(z^n + \sum_{i=1}^n a_i z^{n-i}) \succ \beta\}$ is an algebraic subvariety of \mathbb{C}^n with normalization isomorphic to $\mathbb{C}^{l(\beta)}$.*

For the proof one has just to regard the finite map $\mathbb{C}^n \rightarrow \mathbb{C}^n$, $u \mapsto ((-1)^i \sigma_i(u))_{i=1, \dots, n}$ ($\sigma_i(u)$ is the i -th symmetric polynomial).

A section $X \in \mathcal{Q}(U)$, $U \subset M$ open, induces via the coefficients of the characteristic polynomial $p_{ch, X \circ}$ of multiplication by X a holomorphic map $U \rightarrow \mathbb{C}^n$. Hence the set $\{q \in U \mid P(p_{ch, X \circ}) \succ \beta\}$ is analytic. The intersection of such analytic sets for a basis of sections in U is $\{q \in U \mid P(q) \succ \beta\}$. \square

We suppose that M is connected. Then there is a unique partition β_0 such that $\{p \in M \mid P(p) = \beta_0\}$ is open in M . The complement $\mathcal{K} := \{p \in M \mid P(p) \neq \beta_0\}$ will be called the *caustic* in M ; this name is motivated by the Lagrange maps (chapters 6, 8, 9) and the hypersurface singularities (chapter 16). The multiplication is generically semisimple iff $\beta_0 = (1, \dots, 1)$.

Proposition 2.4. *If the multiplication is generically semisimple then the caustic \mathcal{K} is a hypersurface or empty.*

Proof: Suppose $\dim(\mathcal{K}, p) \leq \dim M - 2$ for some point $p \in \mathcal{K}$. Then in a neighborhood U the complement $U - \mathcal{K}$ is simply connected. There is no monodromy for the locally defined idempotent sections e_1, \dots, e_n on $U - \mathcal{K}$. They are sections on $U - \mathcal{K}$ and extend to a basis of sections on U , with multiplication $e_i \circ e_j = \delta_{ij} e_i$. Hence $\mathcal{K} \cap U = \emptyset$. \square

3. DEFINITION OF F-MANIFOLDS

An F-manifold is a manifold M with a multiplication on the tangent bundle TM which harmonizes with the Lie bracket in the most natural way. A neat formulation of this property requires the Lie derivative of tensors.

Remark 3.1. Here the sheaf of (k, l) -tensors ($k, l \in \mathbb{N}_0$) on a manifold M is the sheaf of \mathcal{O}_M -module homomorphisms $\text{Hom}_{\mathcal{O}_M}(\bigotimes_{i=1}^k \mathcal{T}_M, \bigotimes_{i=1}^l \mathcal{T}_M)$. A $(0, l)$ -tensor $T : \mathcal{O}_M \rightarrow \bigotimes_{i=1}^l \mathcal{T}_M$ can be identified with $T(1)$. Vector fields are $(0, 1)$ -tensors, a (commutative) multiplication on \mathcal{T}_M is a (symmetric) $(2, 1)$ -tensor.

The Lie derivative Lie_X with respect to a vector field X is an \mathcal{O}_M -derivation on the sheaf of (k, l) -tensors. It is $\text{Lie}_X(f) = X(f)$ for functions f , $\text{Lie}_X(Y) = [X, Y]$ for vector fields Y , $\text{Lie}_X(Y_1 \otimes \dots \otimes Y_l) = \sum_i Y_1 \otimes \dots \otimes [X, Y_i] \otimes \dots \otimes Y_l$ for $(0, l)$ -tensors, and $(\text{Lie}_X T)(Y) = \text{Lie}_X(T(Y)) - T(\text{Lie}_X(Y))$ for (k, l) -tensors T . One can always write it explicitly with Lie brackets. Because of the Jacobi identity the Lie derivative satisfies $\text{Lie}_{[X, Y]} = [\text{Lie}_X, \text{Lie}_Y]$.

Definition 3.2. a) An *F-manifold* is a triple (M, \circ, e) where M is a complex connected manifold of dimension ≥ 1 , \circ is a commutative and associative \mathcal{O}_M -bilinear multiplication $\mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{T}_M$, e is a global unit field, and the multiplication satisfies for any two local vector fields X, Y

$$\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ) . \quad (3.1)$$

b) Let (M, \circ, e) be an F-manifold. An *Euler field* E of weight $d \in \mathbb{C}$ is a global vector field E which satisfies

$$\text{Lie}_E(\circ) = d \cdot \circ . \quad (3.2)$$

Remarks 3.3. i) We do not require that the algebras $(T_p M, \circ, e(p))$ are Frobenius algebras (cf. chapter 1). Nevertheless, this is a distinguished class. Frobenius manifolds are F-manifolds [HM][Man](1.5). This will be discussed in chapter 5.

ii) Definition 3.2 differs slightly from the definition of F-manifolds in [HM] by the addition of a global unit field e . This unit field is important, for example, for the definition of $\text{Specan}(\mathcal{T}_M)$. Also, the Euler fields were called weak Euler fields in [HM] in order to separate them from the Euler fields with stronger properties of Frobenius manifolds. This is not necessary here.

iii) (3.1) is equivalent to

$$\begin{aligned} & [X \circ Y, Z \circ W] - [X \circ Y, Z] \circ W - [X \circ Y, W] \circ Z \\ & - X \circ [Y, Z \circ W] + X \circ [Y, Z] \circ W + X \circ [Y, W] \circ Z \\ & - Y \circ [X, Z \circ W] + Y \circ [X, Z] \circ W + Y \circ [X, W] \circ Z = 0 \end{aligned} \quad (3.3)$$

for any four (local) vector fields X, Y, Z, W . (3.2) is equivalent to

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] - d \cdot X \circ Y = 0 \quad (3.4)$$

for any two (local) vector fields X, Y . The left hand sides of (3.3) and (3.4) are \mathcal{O}_M -polylinear with respect to X, Y, Z, W resp. X, Y . Hence they define a $(4, 1)$ - resp. $(2, 1)$ -tensor. In order to check (3.1) and (3.2) for a manifold with multiplication, it suffices to check (3.3) and (3.4) for a basis of vector fields.

iv) The unit field e in an F-manifold (M, \circ, e) plays a distinguished role. It is automatically nowhere vanishing. It is an Euler field of weight 0,

$$\text{Lie}_e(\circ) = 0 \cdot \circ, \quad (3.5)$$

because of (3.1) for $X = Y = e$. So, the multiplication of the F-manifold is constant along the unit field.

v) An Euler field E of weight $d \neq 0$ is not constant along the unit field. But one has for any $d \in \mathbb{C}$

$$[e, E] = d \cdot e, \quad (3.6)$$

because of (3.4) for $X = Y = e$. More generally, in [HM][Man](I.5) the identity

$$[E^{\circ n}, E^{\circ m}] = d(m - n) \cdot E^{\circ(m+n-1)} \quad (3.7)$$

is proved. Chapter 6 will show how intrinsic the notion of an Euler field is for an F-manifold.

vi) The sheaf of Euler fields of an F-manifold (M, \circ, e) is a Lie subalgebra of \mathcal{T}_M . If E_1 and E_2 are Euler fields of weight d_1 and d_2 , then $c \cdot E_1$ ($c \in \mathbb{C}$) is an Euler field of weight $c \cdot d_1$, $E_1 + E_2$ is an Euler field of weight $d_1 + d_2$, and $[E_1, E_2]$ is an Euler field of weight 0. The last holds because of $\text{Lie}_{[E_1, E_2]} = [\text{Lie}_{E_1}, \text{Lie}_{E_2}]$ (cf. Remark 3.1 and [HM][Man](I.5)).

vii) The caustic \mathcal{K} of an F-manifold is the subvariety of points $p \in M$ such that the eigenspace decomposition of $T_p M$ has fewer components than for generic points (cf. chapter 2). The caustic is invariant with respect to e because of (3.5).

4. DECOMPOSITION OF F-MANIFOLDS AND EXAMPLES

Proposition 4.1. *The product of two F-manifolds (M_1, \circ_1, e_1) and (M_2, \circ_2, e_2) is an F-manifold $(M, \circ, e) = (M_1 \times M_2, \circ_1 \oplus \circ_2, e_1 + e_2)$.*

If E_1 and E_2 are Euler fields on M_1 and M_2 of the same weight d then the sum $E_1 + E_2$ (of the lifts to M) is an Euler field of weight d on M .

Proof: $\mathcal{T}_M = \mathcal{O}_M \cdot pr_1^{-1}\mathcal{T}_{M_1} \oplus \mathcal{O}_M \cdot pr_2^{-1}\mathcal{T}_{M_2}$. Vector fields $X_i, Y_i \in pr_i^{-1}\mathcal{T}_{M_i}$, $i = 1, 2$, satisfy

$$\begin{aligned} X_i \circ Y_i &\in pr_i^{-1}\mathcal{T}_{M_i}, \quad [X_i, Y_i] \in pr_i^{-1}\mathcal{T}_{M_i}, \\ X_1 \circ Y_2 &= 0, \quad [X_1, Y_2] = 0. \end{aligned}$$

This together with (3.3) for vector fields in \mathcal{T}_{M_1} and for vector fields in \mathcal{T}_{M_2} gives (3.3) for vector fields in $pr_1^{-1}\mathcal{T}_{M_1} \cup pr_2^{-1}\mathcal{T}_{M_2}$. Because of the \mathcal{O}_M -polylinearity then (3.3) holds for all vector fields. For the same reasons, $E_1 + E_2$ satisfies (3.4). \square

Theorem 4.2. *Let (M, p) be the germ in $p \in M$ of an F -manifold (M, \circ, e) .*

Then the eigenspace decomposition $T_p M = \bigoplus_{k=1}^l (T_p M)_k$ of the algebra $T_p M$ extends to a unique decomposition

$$(M, p) = \prod_{k=1}^l (M_k, p)$$

of the germ (M, p) into a product of germs of F -manifolds. These germs (M_k, p) are irreducible germs of F -manifolds, as already the algebras $T_p M_k \cong (T_p M)_k$ are irreducible.

An Euler field E on (M, p) decomposes into a sum of Euler fields of the same weights on the germs (M_k, p) of F -manifolds.

Proof: By Corollary 2.2 the eigenspace decomposition of $T_p M$ extends in some neighborhood of p to a decomposition of the tangent bundle into a sum of multiplication invariant subbundles. First we have to show that these subbundles and any sum of them are integrable.

Let $\mathcal{T}_{M,p} = \bigoplus_{k=1}^l (\mathcal{T}_{M,p})_k$ be the according decomposition of $\mathcal{T}_{M,p}$ into multiplication invariant free $\mathcal{O}_{M,p}$ -submodules, and $e = \sum_k e_k$ with $e_k \in (\mathcal{T}_{M,p})_k$. Then $e_k \circ : \mathcal{T}_{M,p} \rightarrow (\mathcal{T}_{M,p})_k$ is the projection; $e_j \circ e_k = \delta_{jk} e_k$.

Claim:

- i) $\text{Lie}_{e_k}(\circ) = 0 \cdot \circ$;
- ii) $[e_j, e_k] = 0$;
- iii) $[e_j, (\mathcal{T}_{M,p})_k] \subset (\mathcal{T}_{M,p})_k$;
- iv) $[(\mathcal{T}_{M,p})_j, (\mathcal{T}_{M,p})_k] \subset (\mathcal{T}_{M,p})_j + (\mathcal{T}_{M,p})_k$.

Proof of the claim: i)

$$\delta_{jk} \cdot \text{Lie}_{e_k}(\circ) = \text{Lie}_{e_j \circ e_k}(\circ) = e_j \circ \text{Lie}_{e_k}(\circ) + e_k \circ \text{Lie}_{e_j}(\circ) .$$

This implies for $j \neq k$ as well as for $j = k$ that $e_j \circ \text{Lie}_{e_k}(\circ) = 0 \cdot \circ$. Thus $\text{Lie}_{e_k}(\circ) = 0 \cdot \circ$.

ii) $0 = \text{Lie}_{e_j}(\circ)(e_k, e_k) = [e_j, e_k \circ e_k] - 2e_k \circ [e_j, e_k]$, hence $[e_j, e_k] \in (\mathcal{T}_{M,p})_k$, so for $j \neq k$ we have $[e_j, e_k] = 0$, for $j = k$ anyway.

iii) Suppose $X = e_k \circ X \in (\mathcal{T}_{M,p})_k$; then $0 = \text{Lie}_{e_j}(\circ)(e_k, X) = [e_j, X] - e_k \circ [e_j, X]$.

iv) Suppose $X \in (\mathcal{T}_{M,p})_j$, $Y \in (\mathcal{T}_{M,p})_k$, $k \neq i \neq j$; then $e_i \circ X = 0$ and

$$\begin{aligned} 0 &= \text{Lie}_{e_i \circ X}(\circ)(e_i, Y) = e_i \circ \text{Lie}_X(\circ)(e_i, Y) \\ &= e_i \circ [X, e_i \circ Y] - e_i \circ [X, e_i] \circ Y - e_i \circ [X, Y] \circ e_i \\ &= -e_i \circ [X, Y] . \end{aligned} \quad (\quad \square)$$

Claim iv) shows that for any k the subbundle with germs of sections $\bigoplus_{j \neq k} (\mathcal{T}_{M,p})_j$ is integrable. According to the Frobenius theorem there is a (germ of a) submersion $f_k : (M, p) \rightarrow (\mathbb{C}^{\dim(T_p M)_k}, 0)$ such that the fibers are the integral manifolds of this subbundle. Then $\bigoplus_k f_k : (M, p) \rightarrow (\mathbb{C}^{\dim M}, 0)$ is an isomorphism.

The submanifolds $(M_k, p) := ((\bigoplus_{j \neq k} f_j)^{-1}(0), p)$ yield the decomposition $(M, p) \cong \prod_{k=1}^l (M_k, p)$ with

$$\mathcal{T}_{M,p} = \bigoplus_k \mathcal{O}_{M,p} \cdot pr_k^{-1} \mathcal{T}_{M_k,p} = \bigoplus_k (\mathcal{T}_{M,p})_k .$$

Claim:

- v) $X \circ Y \in pr_k^{-1} \mathcal{T}_{M_k,p}$ if $X, Y \in pr_k^{-1} \mathcal{T}_{M_k,p}$.
- vi) $e_k \circ E \in pr_k^{-1} \mathcal{T}_{M_k,p}$ if E is an Euler field.

Proof of the claim: v) $X \circ Y \in \mathcal{O}_{M,p} \cdot pr_k^{-1} \mathcal{T}_{M_k,p}$ because this is multiplication invariant. Now $X \circ Y \in pr_k^{-1} \mathcal{T}_{M_k,p}$ is true if and only if $[Z, X \circ Y] \in (\mathcal{T}_{M,p})_j$ for any j and any $Z \in (\mathcal{T}_{M,p})_j$; but

$$\begin{aligned} [Z, X \circ Y] &= \text{Lie}_Z(X \circ Y) = \text{Lie}_{e_j \circ Z}(X \circ Y) \\ &= e_j \circ \text{Lie}_Z(X \circ Y) \in (\mathcal{T}_{M,p})_j . \end{aligned}$$

vi) Analogously, for any $Z \in (\mathcal{T}_{M,p})_j$

$$\begin{aligned} &-[Z, e_k \circ E] \\ &= \text{Lie}_{e_k \circ E}(e_j \circ Z) \\ &= \text{Lie}_{e_k \circ E}(\circ)(e_j, Z) + \text{Lie}_{e_k \circ E}(e_j) \circ Z + \text{Lie}_{e_k \circ E}(Z) \circ e_j \\ &= e_k \circ e_j \circ Z + \text{Lie}_{e_k \circ E}(e_j) \circ Z + \text{Lie}_{e_k \circ E}(Z) \circ e_j \in (\mathcal{T}_{M,p})_j . \end{aligned} \quad (\quad \square)$$

Claim v) and vi) show that the multiplication on (M, p) and an Euler field E come from multiplication and vector fields on the submanifolds

(M_k, p) via the decomposition. These satisfy (3.3) and (3.4): this is just the restriction of (3.3) and (3.4) to $pr_k^{-1}\mathcal{T}_{M_k, p}$. \square

Examples 4.3. i) $M = \mathbb{C}$ with coordinate u and unit field $e = \frac{\partial}{\partial u}$ with multiplication $e \circ e = e$ is an F-manifold. The field $E = u \cdot e = u \frac{\partial}{\partial u}$ is an Euler field of weight 1. The space of all Euler fields of weight d is $d \cdot E + \mathbb{C} \cdot e$. One has just to check (3.3) and (3.4) for $X = Y = Z = W = e$ and compare (3.6).

Any 1-dimensional F-manifold is locally isomorphic to an open subset of this F-manifold (\mathbb{C}, \circ, e) . It will be called A_1 .

ii) From i) and Proposition 4.1 one obtains the F-manifold $A_1^n = (\mathbb{C}^n, \circ, e)$ with coordinates u_1, \dots, u_n , idempotent vector fields $e_i = \frac{\partial}{\partial u_i}$, semisimple multiplication $e_i \circ e_j = \delta_{ij} e_i$, unit field $e = \sum_i e_i$ and an Euler field $E = \sum_i u_i \cdot e_i$ of weight 1. Because of Theorem 4.2, the space of Euler fields of weight d is $d \cdot E + \sum_i \mathbb{C} \cdot e_i$.

Also because of Theorem 4.2, any F-manifold M with semisimple multiplication is locally isomorphic to an open subset of the F-manifold A_1^n . The induced local coordinates u_1, \dots, u_n on M are unique up to renumbering and shift. They are called canonical coordinates, following Dubrovin. They are the eigenvalues of a locally defined Euler field of weight 1.

iii) Any complex Frobenius manifold is an F-manifold [HM][Man](I.5).

iv) Especially, the complex orbit space of a finite Coxeter group carries the structure of a Frobenius manifold [Du1][Du2](Lecture 4). The F-manifold structure will be discussed in chapter 18, the Frobenius manifold structure in chapter 19. Here we only give the multiplication and the Euler fields for the 2-dimensional F-manifolds $I_2(m)$, $m \geq 2$, with $I_2(2) = A_1^2$, $I_2(3) = A_2$, $I_2(4) = B_2 = C_2$, $I_2(5) =: H_2$, $I_2(6) = G_2$.

The manifold is $M = \mathbb{C}^2$ with coordinates t_1, t_2 ; we denote $\delta_i := \frac{\partial}{\partial t_i}$. Unit field e and multiplication \circ are given by $e = \delta_1$ and $\delta_2 \circ \delta_2 = t_2^{m-2} \cdot \delta_1$. An Euler field E of weight 1 is $E = t_1 \delta_1 + \frac{2}{m} t_2 \delta_2$. The space of global Euler fields of weight d is $d \cdot E + \mathbb{C} \cdot e$ for $m \geq 3$. The caustic is $\mathcal{K} = \{t \in M \mid t_2 = 0\}$ for $m \geq 3$ and $\mathcal{K} = \emptyset$ for $m = 2$. The multiplication is semisimple outside of \mathcal{K} ; the germ (M, t) is an irreducible germ of an F-manifold iff $t \in \mathcal{K}$. One can check all of this by hand. We will come back to it in Theorem 12.1, when more general results allow more insight.

v) Another 2-dimensional F-manifold is \mathbb{C}^2 with coordinates t_1, t_2 , unit field $e = \delta_1$ and multiplication \circ given by $\delta_2 \circ \delta_2 = 0$. Here all germs (M, t) are irreducible and isomorphic. $E_1 := t_1 \delta_1$ is an Euler field

of weight 1. Contrary to the cases above with generically semisimple multiplication, here the space of Euler fields of weight 0 is infinite dimensional, by (3.4):

$$\begin{aligned}
& \{E \mid \text{Lie}_E(\circ) = 0 \cdot \circ\} \\
&= \{E \mid [\delta_1, E] = 0, \delta_2 \circ [\delta_2, E] = 0\} \\
&= \{\varepsilon_1 \delta_1 + \varepsilon_2(t_2) \delta_2 \mid \varepsilon_1 \in \mathbb{C}, \varepsilon_2 \in \mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2), \delta_1(\varepsilon_2) = 0\} .
\end{aligned} \tag{4.1}$$

vi) The base space of a semiuniversal unfolding of an isolated hypersurface singularity is (a germ of) an F-manifold. The multiplication was defined first by K. Saito [SaK4](1.5) [SaK5](1.3). A good part of the geometry of F-manifolds that will be developed in the next chapters is classical in the case of hypersurface singularities, from different points of view. We will discuss this in chapter 16.

vii) Also the base of a semiuniversal unfolding of a boundary singularity is (a germ of) an F-manifold, compare chapter 17. There are certainly more classes of semiuniversal unfoldings which carry the structure of F-manifolds.

5. F-MANIFOLDS AND POTENTIALITY

The integrability condition (3.1) for the multiplication in F-manifolds and the potentiality condition in Frobenius manifolds are closely related. For semisimple multiplication this has been known previously (with 6.2 i) \iff ii) and e.g. [Hi](Theorem 3.1)). Here we give a general version, requiring neither semisimple multiplication nor flatness of the metric.

Remark 5.1. Let M be a manifold with a connection ∇ . The covariant derivative $\nabla_X T$ of a (k, l) -tensor with respect to a vector field is defined exactly as the Lie derivative $\text{Lie}_X T$ in Remark 3.1., starting with the covariant derivatives of vector fields. ∇_X is a \mathcal{O}_M -derivation on the sheaf of (k, l) -tensors just as Lie_X . But ∇_X is also \mathcal{O}_M -linear in X , opposite to Lie_X . Therefore ∇T is a $(k + 1, l)$ -tensor.

Theorem 5.2. *Let (M, \circ, ∇) be a manifold M with a commutative and associative multiplication \circ on TM and with a torsion free connection ∇ . By definition, $\nabla \circ (X, Y, Z)$ is symmetric in Y and Z .*

If the $(3, 1)$ -tensor $\nabla \circ$ is symmetric in all three arguments, then the multiplication satisfies for any local vector fields X and Y

$$\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ) . \tag{5.1}$$

Proof: $\nabla \circ (X, Y, Z) = \nabla_X(Y \circ Z) - \nabla_X(Y) \circ Z - Y \circ \nabla_X(Z)$ is symmetric in Y and Z . The $(4, 1)$ -tensor

$$\begin{aligned} & (X, Y, Z, W) \\ \mapsto & \nabla \circ (X, Y \circ Z, W) + W \circ \nabla \circ (X, Y, Z) \\ = & \nabla_X(Y \circ Z \circ W) - \nabla_X(Y) \circ Z \circ W \\ & - Y \circ \nabla_X(Z) \circ W - Y \circ Z \circ \nabla_X(W) \end{aligned} \quad (5.2)$$

is symmetric in Y, Z, W . A simple calculation using the torsion freeness of ∇ shows

$$\begin{aligned} & (\text{Lie}_{X \circ Y}(\circ) - X \circ \text{Lie}_Y(\circ) - Y \circ \text{Lie}_X(\circ))(Z, W) \\ = & \nabla \circ (X \circ Y, Z, W) - X \circ \nabla \circ (Y, Z, W) - Y \circ \nabla \circ (X, Z, W) \\ - & \nabla \circ (Z \circ W, X, Y) + Z \circ \nabla \circ (W, X, Y) + W \circ \nabla \circ (Z, X, Y) . \end{aligned} \quad (5.3)$$

If $\nabla \circ$ is symmetric in all three arguments then

$$\nabla \circ (X \circ Y, Z, W) + Z \circ \nabla \circ (W, X, Y) + W \circ \nabla \circ (Z, X, Y) \quad (5.4)$$

is symmetric in X, Y, Z, W because of the symmetry of the tensor in (5.2). Then the right hand side of (5.3) vanishes. \square

Theorem 5.3. *Let (M, \circ, e, g) be a manifold with a commutative and associative multiplication \circ on TM , a unit field e , and a metric g (a symmetric nondegenerate bilinear form) on TM which is multiplication invariant, i.e. the $(3, 0)$ -tensor A ,*

$$A(X, Y, Z) := g(X, Y \circ Z) , \quad (5.5)$$

is symmetric in all three arguments. ∇ denotes the Levi-Civita connection of the metric. The coidentity ε is the 1-form which is defined by $\varepsilon(X) = g(X, e)$. The following conditions are equivalent:

- i) (M, \circ, e) is an F -manifold and ε is closed.*
- ii) The $(4, 0)$ -tensor ∇A is symmetric in all four arguments.*
- iii) The $(3, 1)$ -tensor $\nabla \circ$ is symmetric in all three arguments.*

Proof: The Levi-Civita connection satisfies $\nabla g = 0$. Therefore

$$\begin{aligned} & \nabla A(X, Y, Z, W) \\ = & Xg(Y, Z \circ W) - g(\nabla_X Y, Z \circ W) \\ & - g(Y, W \circ \nabla_X Z) - g(Y, Z \circ \nabla_X W) \\ = & g(Y, \nabla_X(Z \circ W) - W \circ \nabla_X Z - Z \circ \nabla_X W) \\ = & g(Y, \nabla \circ (X, Z, W)) . \end{aligned} \quad (5.6)$$

g is nondegenerate and $\nabla A(X, Y, Z, W)$ is always symmetric in Y, Z, W . Equation (5.6) shows $ii) \iff iii)$. Because of $\nabla g = 0$ and the torsion

freeness $\nabla_X Y - \nabla_Y X = [X, Y]$, the 1-form ε satisfies

$$\begin{aligned} d\varepsilon(X, Y) &= X(\varepsilon(Y)) - Y(\varepsilon(X)) - \varepsilon([X, Y]) \\ &= g(Y, \nabla_X e) - g(X, \nabla_Y e) \\ &= -\nabla A(X, Y, e, e) + \nabla A(Y, X, e, e) . \end{aligned} \quad (5.7)$$

Hence $ii) \Rightarrow d\varepsilon = 0$; with Theorem 5.2 this gives $ii) \Rightarrow i)$. It rests to show $i) \Rightarrow ii)$.

(5.8) and (5.9) follow from the definition of $\nabla \circ$ and from (5.6),

$$\nabla \circ (X, Y, e) = Y \circ \nabla \circ (X, e, e) , \quad (5.8)$$

$$\nabla A(X, U, Y, e) = \nabla A(X, U \circ Y, e, e) . \quad (5.9)$$

One calculates with (5.3) and (5.6)

$$\begin{aligned} &g(e, (\text{Lie}_{X \circ Y}(\circ) - X \circ \text{Lie}_Y(\circ) - Y \circ \text{Lie}_X(\circ))(Z, W)) \quad (5.10) \\ &= \nabla A(X \circ Y, e, Z, W) - \nabla A(Y, X, Z, W) - \nabla A(X, Y, Z, W) \\ &- \nabla A(Z \circ W, e, X, Y) + \nabla A(W, Z, X, Y) + \nabla A(Z, W, X, Y) . \end{aligned}$$

If $i)$ holds then (5.7), (5.9), and (5.10) imply

$$\begin{aligned} &\nabla A(Y, X, Z, W) - \nabla A(W, Z, X, Y) \\ &= -\nabla A(X, Y, Z, W) + \nabla A(Z, W, X, Y) . \end{aligned} \quad (5.11)$$

The left hand side is symmetric in X and Z , the right hand side is skewsymmetric in X and Z , so both sides vanish. ∇A is symmetric in all four arguments. \square

Lemma 5.4. *Let (M, g, ∇) be a manifold with metric g and Levi-Civita connection ∇ . Then a vector field Z is flat, i.e. $\nabla Z = 0$, iff $\text{Lie}_Z(g) = 0$ and the 1-form $\varepsilon_Z := g(Z, \cdot)$ is closed.*

Proof: ∇ is torsion free and satisfies $\nabla g = 0$. Therefore (cf. (5.7))

$$d\varepsilon_Z(X, Y) = g(Y, \nabla_X Z) - g(X, \nabla_Y Z) , \quad (5.12)$$

$$\text{Lie}_Z(g)(X, Y) = g(Y, \nabla_X Z) + g(X, \nabla_Y Z) . \quad (5.13)$$

\square

Remarks 5.5. a) Let (M, \circ, e, g) satisfy the hypotheses of Theorem 5.3 and let g be flat. Then condition $ii)$ in Theorem 5.3 is equivalent to the existence of a local potential $\Phi \in \mathcal{O}_{M,p}$ (for any $p \in M$) with $(XYZ)\Phi = A(X, Y, Z)$ for any flat local vector fields X, Y, Z .

b) In view of this the conditions $ii)$ and $iii)$ in Theorem 5.3 are called *potentiality* conditions.

c) (M, \circ, e, E, g) is a Frobenius manifold if it satisfies the hypotheses and conditions in Theorem 5.3, if g is flat, if $\text{Lie}_e(g) = 0$ (resp. e is flat),

and if E is an Euler field (with respect to M as F-manifold, preferably of weight 1), with $\text{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$.

6. LAGRANGE PROPERTY OF MASSIVE F-MANIFOLDS

Consider an n -dimensional manifold (M, \circ, e) with commutative and associative multiplication on the tangent bundle and with unit field e . Its analytic spectrum $L := \text{Specan}(\mathcal{T}_M)$ is a subvariety of the cotangent bundle T^*M . The cotangent bundle carries a natural symplectic structure, given by the 2-form $d\alpha$. Here α is the canonical 1-form, which is written as $\alpha = \sum_i y_i dt_i$ in local coordinates t_1, \dots, t_n for the base and dual coordinates y_1, \dots, y_n for the fibers $(\mathcal{T}_M \rightarrow (\pi_{T^*M})_* \mathcal{O}_{T^*M}, \frac{\partial}{\partial t_i} \mapsto y_i)$.

The isomorphism $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ from (2.2) can be expressed with α by

$$\mathbf{a}(X) = \alpha(\tilde{X})|_L, \quad (6.1)$$

where $X \in \mathcal{T}_M$ and \tilde{X} is any lift of X to a neighborhood of L in T^*M .

The values of the function $\mathbf{a}(X)$ on $\pi^{-1}(p)$ are the eigenvalues of $X \circ$ on $T_p M$; this follows from Theorem 2.1 and Lemma 1.1.

Definition 6.1. A manifold (M, \circ, e) with commutative and associative multiplication on the tangent bundle and with unit field e is *massive* if the multiplication is generically semisimple.

Then the set of points where the multiplication is not semisimple is empty or a hypersurface, the caustic \mathcal{K} (Proposition 2.4). In the rest of the paper we will study the local structure of massive F-manifolds at points where the multiplication is not semisimple.

It is known that the analytic spectrum of a massive Frobenius manifold is Lagrange (compare [Au] and the references cited there). Theorem 6.2 together with Theorem 5.3 makes the relations between the different conditions transparent.

Theorem 6.2. *Let (M, \circ, e) be a massive n -dimensional manifold M .*

*The analytic spectrum $L = \text{Specan}(\mathcal{T}_M) \subset T^*M$ is an everywhere reduced subvariety. The map $\pi : L \rightarrow M$ is finite and flat. It is a branched covering of degree n , branched above the caustic \mathcal{K} . The following conditions are equivalent.*

- i) (M, \circ, e) is a massive F-manifold;*
- ii) At any semisimple point $p \in M - \mathcal{K}$, the idempotent vector fields $e_1, \dots, e_n \in \mathcal{T}_{M,p}$ commute.*
- iii) $L \subset T^*M$ is a Lagrange variety, i.e. $\alpha|_{L_{\text{reg}}}$ is closed.*

Proof: $L = \pi^{-1}(\mathcal{K})$ is smooth, $\pi : L = \pi^{-1}(\mathcal{K}) \rightarrow M = \mathcal{K}$ is a covering. $\pi_*\mathcal{O}_L (\cong \mathcal{T}_M)$ is a free \mathcal{O}_M -module, so a Cohen–Macaulay \mathcal{O}_M -module and a Cohen–Macaulay ring. Then L is Cohen–Macaulay and everywhere reduced, as it is reduced at generic points (cf. [Lo2] pp 49–51).

It rests to show the equivalences i) \iff ii) \iff iii).

i) \Rightarrow ii) follows from Theorem 4.2 and has been discussed in Example 4.3 ii).

ii) \Rightarrow i) is clear because (3.4) holds everywhere if it holds at generic points (in fact, one point suffices).

ii) \Rightarrow iii) We fix canonical coordinates u_i with $\frac{\partial}{\partial u_i} = e_i$ on (M, p) for a point $p \in M = \mathcal{K}$ and the dual coordinates x_i on the fibers of the cotangent bundle $(\mathcal{T}_M \rightarrow (\pi_{T^*M})_*\mathcal{O}_{T^*M}, e_i \mapsto x_i)$. Then locally above (M, p) the analytic spectrum L is in these coordinates

$$\begin{aligned} L &\cong \{(x_j, u_j) \mid x_1 + \dots + x_n = 1, x_i x_j = \delta_{ij} x_j\} \\ &= \bigcup_{i=1}^n \{(x_j, u_j) \mid x_j = \delta_{ij}\} . \end{aligned} \quad (6.2)$$

The 1-form $\alpha = \sum x_i du_i$ is closed in $L = \pi^{-1}(\mathcal{K})$. This is open and dense in L_{reg} , hence L is a Lagrange variety.

iii) \Rightarrow ii) Above a small neighborhood U of $p \in M = \mathcal{K}$ the analytic spectrum consists of n smooth components L_k , $k = 1, \dots, n$, with $\pi : L_k \xrightarrow{\cong} U$. An idempotent vector field e_i can be lifted to vector fields \tilde{e}_i in neighborhoods U_k of L_k in T^*M such that they are tangent to all L_k . The commutator $[\tilde{e}_i, \tilde{e}_j]$ is a lift of the commutator $[e_i, e_j]$ in these neighborhoods U_k .

$$\begin{aligned} \mathbf{a}([e_i, e_j]|_U) &= \bigcup_k (\alpha([\tilde{e}_i, \tilde{e}_j]))|_{L_k} \\ &= \bigcup_k (\tilde{e}_i(\alpha(\tilde{e}_j)) - \tilde{e}_j(\alpha(\tilde{e}_i)) - d\alpha(\tilde{e}_i, \tilde{e}_j))|_{L_k} \\ &= \bigcup_k (\tilde{e}_i|_{L_k}(\delta_{jk}) - \tilde{e}_j|_{L_k}(\delta_{ik}) - d\alpha|_{L_k}(\tilde{e}_i|_{L_k}, \tilde{e}_j|_{L_k})) = 0 . \end{aligned}$$

But $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_*\mathcal{O}_L$ is an isomorphism, so $[e_i, e_j] = 0$ □

Theorem 6.3. a) Let (M, \circ, e) be a massive F -manifold.

A vector field E is an Euler field of weight $c \in \mathbb{C}$ iff

$$d(\mathbf{a}(E))|_{L_{reg}} = c \cdot \alpha|_{L_{reg}} . \quad (6.3)$$

b) Let $(M, p) = \prod_{k=1}^l (M_k, p)$ be the decomposition of the germ of a massive F -manifold into irreducible germs of F -manifolds (M_k, \circ, e_k) .

i) The space of (germs of) Euler fields of weight 0 for (M, p) is the abelian Lie algebra $\sum_{k=1}^l \mathbb{C} \cdot e_k$.

ii) There is a unique continuous function $F : (L, \pi^{-1}(p)) \rightarrow (\mathbb{C}, 0)$ on the multigerms $(L, \pi^{-1}(p))$ which has value 0 on $\pi^{-1}(p)$, is holomorphic on L_{reg} and satisfies $(dF)|_{L_{reg}} = \alpha|_{L_{reg}}$.

iii) An Euler field of weight $c \neq 0$ for (M, p) exists iff this function F is holomorphic. In that case, $c \cdot \mathbf{a}^{-1}(F)$ is an Euler field of weight c and $\mathbb{C} \cdot \mathbf{a}^{-1}(F) + \sum_{k=1}^l \mathbb{C} \cdot e_k$ is the Lie algebra of all Euler fields on the germ (M, p) .

Proof: a) It is sufficient to prove this locally for a germ (M, p) with $p \in M - \mathcal{K}$. This germ is isomorphic to A_1^n . A vector field $E = \sum_{i=1}^n \varepsilon_i e_i$, $\varepsilon_i \in \mathcal{O}_{M,p}$, is an Euler field of weight c iff $d\varepsilon_i = c \cdot du_i$ (Theorem 4.2 and Example 4.3 ii)).

Going into the proof of 6.2 ii) \Rightarrow iii), one sees that this is equivalent to (6.3).

b) The multigerms $(L, \pi^{-1}(p))$ has l components, the space of locally constant functions on it has dimension l . The function (multigerms) F exists because $\alpha|_{L_{reg}}$ is closed. This will be explained in the next chapter (Lemma 7.1). All statements follow now with a). \square

7. EXISTENCE OF EULER FIELDS

By Theorem 6.2, the analytic spectrum (L, λ) of an irreducible germ (M, p) of a massive F-manifold is a germ of an (often singular) Lagrangian variety, and $(L, \lambda) \hookrightarrow (T^*M, \lambda) \rightarrow (M, p)$ is a germ of a Lagrangian map. The paper [Gi2] of Givental is devoted to such objects. It contains implicitly many results on massive F-manifolds. It will be extremely useful and often cited in the following.

The question when a germ of a massive F-manifold has an Euler field of weight 1 is reduced by Theorem 6.3 b)iii) to the question when the function germ F there is holomorphic. Partial answers are given in Corollary 7.5 and Lemma 7.6. We start with a more general situation, as in [Gi2](chapter 1.1).

Let $(L, 0) \subset (\mathbb{C}^N, 0)$ be a reduced complex space germ. Statements on germs will often be formulated using representatives, but they are welldefined for the germs, e.g. " $\alpha|_{L_{reg}}$ is closed" for $\alpha \in \Omega_{\mathbb{C}^N, 0}^k$.

Lemma 7.1. *Let $\alpha \in \Omega_{\mathbb{C}^N, 0}^1$ be closed on L_{reg} . Then there exists a unique function germ $F : (L, 0) \rightarrow (\mathbb{C}, 0)$ which is holomorphic on L_{reg} , continuous on L and satisfies $dF|_{L_{reg}} = \alpha|_{L_{reg}}$.*

Proof: $(L, 0)$ is homeomorphic to a cone as it admits a Whitney stratification. One can integrate α along paths corresponding to such

a cone structure, starting from 0. One obtains a continuous function F on L , which is holomorphic on L_{reg} because of $d\alpha|_{L_{reg}} = 0$ and which satisfies $dF|_{L_{reg}} = \alpha|_{L_{reg}}$. The unicity of F with value $F(0) = 0$ is clear. \square

Which germs $(L, 0)$ have the property that all such function germs are holomorphic on $(L, 0)$? This property has not been studied much. It can be seen in a line with normality and maximality of complex structures and is weaker than maximality.

It can be rephrased as $H_{Giv}^1((L, 0)) = 0$. Here $H_{Giv}^*(L, 0)$ is the cohomology of the de Rham complex

$$(\Omega_{\mathbb{C}^N, 0}^* / \{\omega \in \Omega_{\mathbb{C}^N, 0}^* \mid \omega|_{L_{reg}} = 0\}, d), \quad (7.1)$$

which is considered in [Gi2](chapter 1.1). We state some known results on this cohomology.

Theorem 7.2. *a) (Poincare-Lemma, [Gi2](chapter 1.1)) If $(L, 0)$ is weighted homogeneous with positive weights then $H_{Giv}^*(L, 0) = 0$.*

b) ([Va]) Suppose that $(L, 0)$ is a germ of a hypersurface with an isolated singularity, $(L, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ and $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is a holomorphic function with an isolated singularity. Then

$$\begin{aligned} \dim H_{Giv}^n((L, 0)) &= \mu - \tau \\ &= \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / \left(\frac{\partial f}{\partial x_i}\right) - \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / \left(f, \frac{\partial f}{\partial x_i}\right). \end{aligned} \quad (7.2)$$

c) (essentially Varchenko and Givental, [Gi2](chapter 1.2)) Let $(L, 0)$ be as in b) with $\mu - \tau \neq 0$. The class $[\eta] \in H_{Giv}^n((L, 0))$ of $\eta \in \Omega_{\mathbb{C}^{n+1}, 0}^n$ is not vanishing if $d\eta$ is a volume form, i.e. $d\eta = h dx_0 \dots dx_n$ with $h(0) \neq 0$.

Remarks 7.3. i) The proofs of b) and c) use the Gauß-Manin connection for isolated hypersurface singularities.

ii) c) was formulated in [Gi2](chapter 1.2) only for $n = 1$. The missing piece for the proof for all n was the following fact, which was at that time only known for $n = 1$: *The exponent of a form $h dx_0 \dots dx_n$ is the minimal exponent iff $h(0) \neq 0$.*

This fact has been established by M. Saito [SaM](3.11) for all n .

iii) By a result of K. Saito [SaK1], an isolated hypersurface singularity $(L, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ is weighted homogeneous (with positive weights) iff $\mu - \tau = 0$.

iv) For us only the case $n = 1$ in Theorem 7.2 b)+c) is relevant. Proposition 7.4, which is also due to Givental, implies the following:

Of all isolated hypersurface singularities $(L, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ only the curve singularities ($n = 1$) turn up as germs of Lagrange varieties. These are, of course, germs of Lagrange varieties with respect to any volume form on $(\mathbb{C}^2, 0)$.

v) If $(L, 0) \subset ((S, 0), \omega)$ is the germ of a Lagrange variety in a symplectic space S with symplectic form ω , then the class $[\alpha] \in H_{Giv}^1((L, 0))$ of some α with $d\alpha = \omega$ is independent of the choice of α . It is called the characteristic class of $(L, 0) \subset ((S, 0), \omega)$.

vi) Givental made the conjecture [Gi2](chapter 1.2): *Let $(L, 0)$ be an n -dimensional Lagrange germ. If $H_{Giv}^n((L, 0)) \neq 0$ then $H_{Giv}^1((L, 0)) \neq 0$ and the characteristic class $[\alpha] \in H_{Giv}^1((L, 0))$ is nonzero.*

It is true for $n = 1$ because of Theorem 7.2 and Remark 7.3 iii). Givental sees the conjecture in analogy with a conjecture of Arnold which was proved by Gromov 1985 (cf. [Gi2](chapter 1.2)): *any real closed Lagrange manifold $L \subset T^*\mathbb{R}^n$ has nonvanishing characteristic class $[\alpha] \in H^1(L, \mathbb{R})$.*

Proposition 7.4. ([Gi2](chapter 1.1)) *An n -dimensional germ $(L, 0)$ of a Lagrange variety with embedding dimension $\text{embdim}(L, 0) = n + k$ is a product of a k -dimensional Lagrange germ $(L', 0)$ with $\text{embdim}(L', 0) = 2k$ and a smooth $(n - k)$ -dimensional Lagrange germ $(L'', 0)$; here the decomposition of $(L, 0)$ corresponds to a decomposition*

$$((S, 0), \omega) \cong ((S', 0), \omega') \times ((S'', 0), \omega'') \quad (7.3)$$

of the symplectic space germ $(S, 0) \supset (L, 0)$.

Proof: If $k < n$ then a holomorphic function f on S exists with smooth fiber $f^{-1}(0) \supset L$. The Hamilton flow of this function f respects L and the fibers of f . The spaces of orbits in $f^{-1}(0)$ and L give a symplectic space germ of dimension $2n - 2$ and in it a Lagrange germ (e.g. [AGV] 18.2).

To obtain a decomposition as in (7.3) one chooses a germ $(\Sigma, 0) \subset (S, 0)$ of a $2n - 1$ -dimensional submanifold Σ in S which is transversal to the Hamilton field H_f of f . There is a unique section v in $(TS)|_\Sigma$ with $\omega(H_f, v) = 1$ and $\omega(T_p\Sigma, v) = 0$ for $p \in \Sigma$. The shift \tilde{v} of v with the Hamilton flow of f forms together with H_f a 2-dimensional integrable distribution on S , because of $0 = \text{Lie}_{H_f}(\tilde{v}) = [H_f, \tilde{v}]$.

This distribution is everywhere complementary and orthogonal to the integrable distribution whose integral manifolds are the intersections of the fibers of f with the shifts of Σ by the Hamilton flow of f . This yields a decomposition $(S, 0) \cong (\mathbb{C}^2, 0) \times (\Sigma \cap f^{-1}(0), 0)$. One can check that the symplectic form decomposes as required. If $k < n - 1$ one repeats this process. \square

Corollary 7.5. *a) Let $(L, 0)$ be an n -dimensional Lagrange germ isomorphic to $(L', 0) \times (\mathbb{C}^{n-1}, 0)$ as complex space germ. Then $(L', 0)$ is a plane curve singularity. The characteristic class $[\alpha] \in H_{Giv}^1((L, 0))$ is vanishing iff $(L', 0)$ is weighted homogeneous.*

*b) Let $(L, \lambda) \subset (T^*M, \lambda)$ be the analytic spectrum of an irreducible germ (M, p) of a massive F-manifold. Suppose $(L, \lambda) \cong (L', 0) \times (\mathbb{C}^{n-1}, 0)$. Then there exists an Euler field of weight 1 on (M, p) iff $(L', 0)$ is weighted homogeneous.*

Proof: a) Proposition 7.4, Theorem 7.2, and Remark 7.3 iii).

b) Part a) and Theorem 6.3 b)iii). □

In Proposition 20.1 for any plane curve singularity $(L', 0)$ irreducible germs of F-manifolds with analytic spectrum $(L, \lambda) \cong (L', 0) \times (\mathbb{C}^{n-1}, 0)$ for some n will be constructed. So, often there exists no Euler field of weight 1 on a germ of a massive F-manifold. On the other hand, the Poincare-Lemma 7.2 a) and Proposition 7.4 say that an Euler field of weight 1 exists on a germ of a massive F-manifold (M, p) if the multigerms $(L, \pi^{-1}(p))$ of the analytic spectrum is at all points of $\pi^{-1}(p)$ a product of a smooth germ and a germ which is weighted homogeneous with positive weights. Also, we have the following.

Lemma 7.6. *Let M be a massive F-manifold and $F : L \rightarrow \mathbb{C}$ a continuous function with $dF|_{L_{reg}} = \alpha|_{L_{reg}}$. Then $\mathbf{a}^{-1}(F|(L - \pi^{-1}(\mathcal{K})))$ is an Euler field of weight 1 on $M - \mathcal{K}$. It extends to an Euler field on M if (L, λ) is at all points $\lambda \in L$ outside of a subset of codimension ≥ 2 a product of a smooth germ and a germ which is weighted homogeneous with positive weights.*

Proof: Suppose $K \subset L$ is a subset of codimension ≥ 2 with this property. Then F is holomorphic in $L - K$ because of the Poincare-Lemma 7.2 a) and Proposition 7.4. The Euler field extends to $M - \pi^{-1}(K)$. But $\pi(K)$ has also codimension ≥ 2 . So the Euler field extends to M (and F is holomorphic on L). □

8. LYASHKO-LOOIJENGA MAPS AND GRAPHS OF LAGRANGE MAPS

In this chapter classical facts on Lagrange maps are presented, close to [Gi2](chapter 1.3), but slightly more general. They will be used in chapters 9 – 11.

Let $L \subset T^*M$ be a Lagrange variety (not necessarily smooth) in the cotangent bundle of an m -dimensional connected manifold M . We assume:

- a) The projection $\pi : L \rightarrow M$ is a branched covering of degree n , that is, there exists a subvariety $K \subset M$ such that $\pi : L - \pi^{-1}(K) \rightarrow M - K$ is a covering of degree n ($\pi : L \rightarrow M$ is not necessarily flat).
- b) There exists a *generating function* $F : L \rightarrow \mathbb{C}$, that is, a continuous function which is holomorphic on L_{reg} with $dF|_{L_{reg}} = \alpha|_{L_{reg}}$ (locally such a function exists by Lemma 7.1).

Such a function F will be fixed. It can be considered as a multivalued function on $M - K$; the 1-graph of this multivalued function is $L - \pi^{-1}(K)$.

The *Lyashko-Looijenga map* $\Lambda = (\Lambda_1, \dots, \Lambda_n) : M \rightarrow \mathbb{C}^n$ of $L \subset T^*M$ and F is defined as follows: for $q \in M - K$, the roots of the unitary polynomial $z^n + \sum_{i=1}^n \Lambda_i(q) z^{n-i}$ are the values of F on $\pi^{-1}(q)$. It extends to a holomorphic map on M because F is holomorphic on L_{reg} and continuous on L .

The *reduced Lyashko-Looijenga map* $\Lambda^{(red)} = (\Lambda_2^{(red)}, \dots, \Lambda_n^{(red)}) : M \rightarrow \mathbb{C}^{n-1}$ of $L \subset T^*M$ and F is defined as follows: for $q \in M - K$, the roots of the unitary polynomial $z^n + \sum_{i=2}^n \Lambda_i^{(red)}(q) z^{n-i}$ are the values of F on $\pi^{-1}(q)$, shifted by their center $-\frac{1}{n}\Lambda_1(q) = \frac{1}{n} \sum_{\lambda \in \pi^{-1}(q)} F(\lambda)$. It also extends to a holomorphic map on M . Its significance will be discussed after the Remarks 8.2.

The *front* Φ_L of $L \subset T^*M$ and F is the image $\text{Im}(F, pr) \subset \mathbb{C} \times M$ of $(F, pr) : L \rightarrow \mathbb{C} \times M$. It is the zero set of the polynomial $z^n + \sum_{i=1}^n \Lambda_i \cdot z^{n-i}$. So, it is an analytic hypersurface even if F is not holomorphic on all of L .

Following Teissier ([Tei](2.4, 5.5), [Lo2](4.C)), the *development* $\widetilde{\Phi}_L \subset \mathbb{P}T^*(\mathbb{C} \times M)$ of this hypersurface Φ_L in $\mathbb{C} \times M$ is defined as the closure in $\mathbb{P}T^*(\mathbb{C} \times M)$ of the set of tangent hyperplanes at the smooth points of Φ_L . It is an analytic subvariety and a Legendre variety with respect to the canonical contact structure on $\mathbb{P}T^*(\mathbb{C} \times M)$.

The map

$$\mathbb{C} \times T^*M \rightarrow \mathbb{P}T^*(\mathbb{C} \times M), (c, \lambda) \mapsto ((dz - \lambda)^{-1}(0), (c, p)) \quad (8.1)$$

$(\lambda \in T_p^*M \text{ and } dz - \lambda \in T_{(c,p)}^*(\mathbb{C} \times M) \cong T_c^*\mathbb{C} \times T_p^*M)$ identifies $\mathbb{C} \times T^*M$ with the open subset in $\mathbb{P}T^*(\mathbb{C} \times M)$ of hypersurfaces in the tangent spaces which do not contain $\mathbb{C} \cdot \frac{\partial}{\partial z}$. The induced contact structure on $\mathbb{C} \times T^*M$ is given by the 1-form $dz - \alpha$.

The following fact is wellknown. It is one form of appearance of the relation between Lagrange and Legendre maps (e.g. [AGV] 18.–20.). To check it, one has to consider F as a multivalued function on $M - K$ and Φ_L as its graph.

Proposition 8.1. *The embedding $\mathbb{C} \times T^*M \hookrightarrow \mathbb{P}T^*(\mathbb{C} \times M)$ identifies the graph $\text{Im}(F, id) \subset \mathbb{C} \times T^*M$ of $F : L \rightarrow \mathbb{C}$ with the development $\widetilde{\Phi}_L$ of the front Φ_L .*

Remarks 8.2. It has some nontrivial consequences.

i) The polynomial $z^n + \sum_{i=1}^n \Lambda_i z^{n-i}$ has no multiple factors and the branched covering $\Phi_L \rightarrow M$ has degree n : over any point $p \in M - K$ the varieties L and $\widetilde{\Phi}_L$ have n points and the points of Φ_L have n tangent planes; so, over a generic point $p \in M - K$ also Φ_L has n points.

ii) The graph $\text{Im}(F, id) \cong \widetilde{\Phi}_L$ is an analytic variety even if F is not holomorphic on all of L .

iii) The composition of maps $\widetilde{\Phi}_L \xrightarrow{\cong} \text{Im}(F, id) \xrightarrow{pr} L$ is a bijective morphism. It is an isomorphism iff F is holomorphic on L . Also, the continuous map $L \rightarrow \Phi_L$ is a morphism iff F is holomorphic on L .

iv) The Lagrange variety $L \subset T^*M$ together with the values of F at one point of each connectivity component of L and any of the following data determine each other uniquely: the front Φ_L , the development $\widetilde{\Phi}_L$, the Lyashko-Looijenga map Λ , the generating function F as a multivalued function on the base M .

To motivate the reduced Lyashko-Looijenga map, we have to talk about Lagrange maps and their isomorphisms ([AGV]18., [Gi2]3.1).

A Lagrange map is a diagram $L \hookrightarrow (S, \omega) \rightarrow M$ where L is a Lagrange variety in a symplectic manifold (S, ω) and $S \rightarrow M$ is a Lagrange fibration. An isomorphism between two Lagrange maps is given by an isomorphism of the Lagrange fibrations which maps one Lagrange variety to the other.

An automorphism of $T^*M \rightarrow M$ as Lagrange fibration which fixes the base is given by a shift in the fibers,

$$T^*M \rightarrow T^*M, \lambda \mapsto \lambda + dS,$$

where $S : M \rightarrow \mathbb{C}$ is holomorphic ([AGV]18.5). So, regarding $T^*M \rightarrow M$ as a Lagrange fibration means to forget the 0-section and the 1-form α , but to keep the Lagrange fibration and the class $\alpha + \{dS \mid S : M \rightarrow \mathbb{C} \text{ holomorphic}\}$ of 1-forms.

Corollary 8.3. *Let $L \hookrightarrow T^*M \rightarrow M$ be as above (satisfying the assumptions a) and b)) with l connectivity components and points $\lambda_1, \dots, \lambda_l$, one in each connectivity component. The data in i) – iii) are equivalent.*

- i) $L \hookrightarrow T^*M \rightarrow M$ as a Lagrange map and the differences $F(\lambda_i) - F(\lambda_j) \in \mathbb{C}$ of values of F ,
- ii) the generating function F modulo addition of a function on the base,
- iii) the reduced Lyashko-Looijenga map $\Lambda^{(red)} : M \rightarrow \mathbb{C}^{n-1}$.

Proof: i) \Rightarrow ii): Integrating the 1-forms in $\alpha + \{dS \mid S : M \rightarrow \mathbb{C} \text{ holomorphic}\}$ gives ii).

ii) \Rightarrow iii) Definition of $\Lambda^{(red)}$.

iii) \Rightarrow i): $(0, \Lambda^{(red)}) = (0, \Lambda_2^{(red)}, \dots, \Lambda_n^{(red)}) : M \rightarrow \mathbb{C}^n$ is the Lyashko-Looijenga map of a Lagrange variety in T^*M which differs from the original Lagrange variety only by the shift of $d(\frac{-1}{n}\Lambda_1)$ in the fibers. $(0, \Lambda^{(red)})$ determines this Lagrange variety and a generating function for it because of Remark 8.2 iv). One recovers i). \square

9. MINIVERSAL LAGRANGE MAPS AND F-MANIFOLDS

The notion of a miniversal germ of a Lagrange map is central in Givental's paper [Gi2]. We need a slight generalization to multigerms, taking a semilocal viewpoint.

Let $L \subset T^*M$ be a Lagrange variety with finite projection $\pi : L \rightarrow M$. The germ at the base point $p \in M$ of $L \hookrightarrow T^*M \rightarrow M$ is the diagram

$$(L, \pi^{-1}(p)) \hookrightarrow (T^*M, T_p^*M) \rightarrow (M, p).$$

Here $(L, \pi^{-1}(p))$ is a multigerm. $(T^*M, T_p^*M) \rightarrow (M, p)$ is the cotangent bundle of the germ (M, p) ; it is a germ in the base, but not in the fiber.

For this diagram the morphisms

$$\mathcal{O}_{M,p} \oplus \mathcal{T}_{M,p} \rightarrow \pi_*(\mathcal{O}_L)_p, \quad (c, X) \mapsto (c + \alpha(\tilde{X}))|_L \quad (9.1)$$

and

$$\begin{aligned} \mathbb{C} \oplus T_p M &\rightarrow \pi_*(\mathcal{O}_L)_p / \mathbf{m}_p \cdot \pi_*(\mathcal{O}_L)_p, \\ (c, X) &\mapsto (c + \alpha(\tilde{X}))|_{\pi_*(\mathcal{O}_L)_p / \mathbf{m}_p \cdot \pi_*(\mathcal{O}_L)_p} \end{aligned} \quad (9.2)$$

are welldefined. Here \tilde{X} is in both cases a lift of X to T^*M . These morphisms are not invariants of the diagram as a germ at the base point $p \in M$ of a Lagrange map because the identification of the Lagrange fibration with the cotangent bundle of (M, p) is unique only up to shifts in the fibers and only the class of 1-forms $\alpha + \{dS \mid S : M \rightarrow \mathbb{C} \text{ holomorphic}\}$ is uniquely determined (cf. chapter 8).

But being an isomorphism or epimorphism in (9.1) and (9.2) are clearly properties of the germ at p of the Lagrange map.

Definition 9.1. The germ at $p \in M$ of $L \hookrightarrow T^*M \rightarrow M$ as a Lagrange map is called *miniversal* (*versal*) if the morphism in (9.2) is an isomorphism (epimorphism) (cf. [Gi2](chapter 1.3)).

We are only interested in the case of a flat projection $\pi : L \rightarrow M$. Wellknown criteria of flatness for finite maps give the next lemma.

Lemma 9.2. *The following conditions are equivalent.*

- i) *The germ at $p \in M$ of $L \hookrightarrow T^*M \rightarrow M$ as a Lagrange map is miniversal with flat projection $\pi : (L, \pi^{-1}(p)) \rightarrow (M, p)$,*
- ii) *it is miniversal with $\deg \pi = 1 + \dim M$,*
- iii) *the morphism in (9.1) is an isomorphism,*
- iv) *the Lagrange map is miniversal at all points in a neighborhood of $p \in M$.*

Example 9.3. A miniversal germ at a base point of a Lagrange map with not flat projection $\pi : L \rightarrow M$ is given by the germ at $0 \in \mathbb{C}^2$ of the Lagrange fibration $\mathbb{C}^4 \rightarrow \mathbb{C}^2$, $(y_2, y_3, t_2, t_3) \mapsto (t_2, t_3)$ with $\omega = dy_2 dt_2 + dy_3 dt_3$ and by the Lagrange variety L which is the union of two appropriate planes and which is defined by the ideal

$$(y_2, y_3) \cap (y_2 - t_2, y_3 - t_3) = (y_2, y_3) \cdot (y_2 - t_2, y_3 - t_3) . \quad (9.3)$$

Now let M be a massive n -dimensional F-manifold with analytic spectrum $L \subset T^*M$. Then $(L, \pi^{-1}(p)) \hookrightarrow (T^*M, T_p^*M) \rightarrow (M, p)$ is for any $p \in M$ a versal, but not a miniversal germ at the base point $p \in M$ of a Lagrange map. But there is a miniversal one.

$pr_e : (M, p) \rightarrow (M^{(r)}, p^{(r)})$ denotes the germ of the fibration at p whose fibers are the orbits of the unit field e . The fiberwise linear function on T^*M which corresponds to e is called y_1 . Its Hamilton field $\tilde{e} := H_{y_1}$ is a lift of e to T^*M . It leaves the hypersurface $y_1^{-1}(1) \subset T^*M$ and the Lagrange variety $L \subset y_1^{-1}(1)$ invariant. The orbits of \tilde{e} in $y_1^{-1}(1)$ form a germ of a $2n - 2$ -dimensional symplectic manifold with a Lagrange fibration, which can be identified with the cotangent bundle

$$(T^*M^{(r)}, T_{p^{(r)}}^*M^{(r)}) \rightarrow (M^{(r)}, p^{(r)}) .$$

But this identification is only unique up to shifts in the fibers. The orbits of \tilde{e} in L form a Lagrange variety $L^{(r)} \subset T^*M^{(r)}$.

The germ at $p^{(r)} \in M^{(r)}$ of the diagram $L^{(r)} \hookrightarrow T^*M^{(r)} \rightarrow M^{(r)}$ is unique up to isomorphism of *germs in the base of Lagrange maps*. It will be called the *restricted Lagrange map* of the germ (M, p) of the F-manifold M . An explicit description will be given in the proof of the next result.

Theorem 9.4. *a) The restricted Lagrange map of the germ (M, p) of a massive F -manifold is miniversal with flat projection $\pi^{(r)} : L^{(r)} \rightarrow M^{(r)}$.*

b) It determines the germ (M, p) of the F -manifold uniquely.

*c) Any miniversal germ at a base point of a Lagrange map $L' \hookrightarrow T^*M' \rightarrow M'$ with flat projection $L' \rightarrow M'$ is the restricted Lagrange map of a germ of a massive F -manifold.*

Proof: a) In order to be as explicit as possible we choose coordinates $t = (t_1, t') = (t_1, \dots, t_n) : (M, p) \rightarrow (\mathbb{C}^n, 0)$ with $e(t_1) = 1$. The dual coordinates on T^*M are $(y_1, \dots, y_n) = (y_1, y') = y$. The multiplication is given by $\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} = \sum_k a_{ij}^k(t') \frac{\partial}{\partial t_k}$ and the analytic spectrum L is

$$L \cong \{(y, t) \in \mathbb{C}^n \times (\mathbb{C}^n, 0) \mid y_1 = 1, y_i y_j = \sum_k a_{ij}^k(t') y_k\}. \quad (9.4)$$

The restricted Lagrange map is represented by the Lagrange fibration

$$\mathbb{C}^{n-1} \times (\mathbb{C}^{n-1}, 0) \rightarrow (\mathbb{C}^{n-1}, 0), (y', t') \mapsto t' \quad (9.5)$$

with canonical 1-form $\alpha' := \sum_{i \geq 2} y_i dt_i$ and by the Lagrange variety

$$\{(y', t') \in \mathbb{C}^{n-1} \times (\mathbb{C}^{n-1}, 0) \mid y_i y_j = a_{ij}^1(t') + \sum_{k \geq 2} a_{ij}^k(t') y_k \text{ for } i, j \geq 2\} \cong L^{(r)}. \quad (9.6)$$

The equations for the Lagrange variety in (9.6) show that the morphism in (9.1) for this Lagrange map with fixed canonical 1-form α' is an isomorphism. This implies that the restricted Lagrange map for (M, p) is miniversal with flat projection.

b)+c) Any miniversal germ at a base point $p' \in M'$ of a Lagrange map $L' \hookrightarrow T^*M' \rightarrow M'$ with flat projection $\pi' : L' \rightarrow M'$ can be represented by a Lagrange fibration as in (9.5) and a Lagrange variety as in (9.6).

Defining L by (9.4) and $M := \mathbb{C} \times (M', p')$ with $e := \frac{\partial}{\partial t_1}$, one obtains an F -manifold with unit field e and analytic spectrum L .

It rests to show that this does not depend on the way in which the Lagrange fibration is identified with the cotangent bundle of $(\mathbb{C}^{n-1}, 0)$ in (9.5). But one sees easily that a shift in the fibers of (9.5) of the type $y_i \mapsto y_i + \frac{\partial S}{\partial t_i}$ for some holomorphic function $S : (\mathbb{C}^{n-1}, 0) \rightarrow \mathbb{C}$ on the base corresponds only to a change of the coordinate fields $\frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_n}$ and the coordinate t_1 in M and thus to a shift of the section $\{0\} \times (M', p')$ in $M \rightarrow (M', p')$. It does not affect L and the multiplication on (M, p) .

□

Let (M, p) be a germ of a massive F-manifold. The germ

$$\mathcal{G}_{p^{(r)}} := \{X \in \mathcal{T}_{M,p} \mid [e, X] = 0\} \quad (9.7)$$

is a free $\mathcal{O}_{M^{(r)}, p^{(r)}}$ -module of rank n . It is an $\mathcal{O}_{M^{(r)}, p^{(r)}}$ -algebra because of $\text{Lie}_e(\circ) = 0 \cdot \circ$. The functions $\mathbf{a}(X)$ for $X \in \mathcal{G}_{p^{(r)}}$ are invariant with respect to \tilde{e} and induce holomorphic functions on $L^{(r)}$. One obtains a map

$$\mathbf{a}^{(r)} : \mathcal{G}_{p^{(r)}} \rightarrow (\pi_*^{(r)} \mathcal{O}_{L^{(r)}})_{p^{(r)}}. \quad (9.8)$$

Lemma 9.5. $\mathbf{a}^{(r)}$ is an isomorphism of $\mathcal{O}_{M^{(r)}, p^{(r)}}$ -algebras.

Proof: The isomorphism $\mathbf{a} : \mathcal{T}_{M,p} \rightarrow (\pi_* \mathcal{O}_L)_p$ maps the e -invariant vector fields in (M, p) to the \tilde{e} -invariant functions in $(\pi_* \mathcal{O}_L)_p$. \square

This isomorphism $\mathbf{a}^{(r)}$ is closely related to (9.1) for the restricted Lagrange map of (M, p) :

An isomorphism as in (9.1) requires the choice of a 1-form for its Lagrange fibration. The choice of a function $t_1 : (M, p) \rightarrow (\mathbb{C}, 0)$ with $e(t_1) = 1$ yields such a 1-form: the 1-form which is induced by $\alpha - dt_1$ ($\alpha - dt_1$ on T^*M is \tilde{e} -invariant and vanishes on \tilde{e} and induces a 1-form on the space of \tilde{e} -orbits of $y_1^{-1}(1)$).

The choice of t_1 also yields an isomorphism

$$\begin{aligned} & \mathcal{O}_{M^{(r)}, p^{(r)}} \oplus \mathcal{T}_{M^{(r)}, p^{(r)}} \\ \rightarrow & \mathcal{O}_{M^{(r)}, p^{(r)}} \cdot e \oplus \{X \in \mathcal{G}_{p^{(r)}} \mid X(t_1) = 0\} = \mathcal{G}_{p^{(r)}}. \end{aligned} \quad (9.9)$$

One sees with the proof of Theorem 9.4 a) that the composition of this isomorphism with $\mathbf{a}^{(r)}$ gives the isomorphism in (9.1) for the restricted Lagrange map of (M, p) (the germ (M, p) in (9.1) is here $(M^{(r)}, p^{(r)})$).

10. LYASHKO-LOOIJENGA MAP OF AN F-MANIFOLD

Definition 10.1. Let (M, \circ, e) be a massive n -dimensional F-manifold with analytic spectrum $L \subset T^*M$.

a) A *generating function* F for (M, \circ, e) is a generating function for L , that is, a continuous function $F : L \rightarrow \mathbb{C}$ which is holomorphic on L_{reg} with $dF|_{L_{reg}} = \alpha|_{L_{reg}}$.

b) Let F be a generating function for (M, \circ, e) .

i) The *bifurcation diagram* $\mathcal{B} \subset M$ of (M, \circ, e, F) is the set of points $p \in M$ such that F has less than n different values on $\pi^{-1}(p)$.

ii) The *Lyashko-Looijenga map* $\Lambda = (\Lambda_1, \dots, \Lambda_n) : M \rightarrow \mathbb{C}^n$ of (M, \circ, e, F) is the Lyashko-Looijenga map of F as generating function for $L \subset T^*M$ (cf. chapter 8).

iii) The *discriminant* $\mathcal{D} \subset M$ of (M, \circ, e, F) is $\mathcal{D} := \Lambda_n^{-1}(0)$.

The discriminant will be discussed in chapter 11. A generating function for an F-manifold exists locally (Lemma 7.1), but not necessarily globally.

A holomorphic generating function F corresponds to an Euler field $E := \mathbf{a}^{-1}(F)$ of weight 1 (Theorem 6.3); then the values of F on $\pi^{-1}(p)$, $p \in M$, are the eigenvalues of $E \circ : T_p M \rightarrow T_p M$. The objects \mathcal{B} , Λ , \mathcal{D} of Definition 10.1 b) are welldefined for (M, \circ, e, E) if E is such an Euler field.

The restriction of $\pi : L \rightarrow M$ to the complement of the caustic \mathcal{K} is a covering $\pi : \pi^{-1}(M - \mathcal{K}) \rightarrow M - \mathcal{K}$ of degree n , and $\pi^{-1}(M - \mathcal{K})$ is smooth. Hence there a generating function F is holomorphic and corresponds to an Euler field E on $M - \mathcal{K}$. Results and examples about the extendability of E to M are given in Lemma 7.6 and Theorem 20.4.

The bifurcation diagram \mathcal{B} of (M, \circ, e, F) contains the caustic \mathcal{K} . The caustic is a hypersurface or empty (Proposition 2.4) and invariant with respect to the unit field e (Remark 3.3 vii)). The bifurcation diagram has the same properties: the restriction of F to an open set $U \subset M - \mathcal{K}$ with canonical coordinates (u_1, \dots, u_n) corresponds to an Euler field $E = \sum (u_i + r_i) e_i$ for some $r_i \in \mathbb{C}$, and the bifurcation diagram is the hypersurface

$$\mathcal{B} \cap U = U \cap \{u \mid u_i + r_i = u_j + r_j \text{ for some } i \neq j\}.$$

It is invariant with respect to e because of $e(u_i - u_j) = 0$.

The Lyashko-Looijenga map for the F-manifold $A_1^n = (\mathbb{C}^n, \circ, e)$ (Example 4.3 ii)) with Euler field $E = \sum u_i e_i$ and Euler field-function $F := \mathbf{a}(E)$ is

$$\Lambda^{(n)} : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad u \mapsto ((-1)^i \sigma_i(u)) \quad (10.1)$$

($\sigma_1(u), \dots, \sigma_n(u)$ are the symmetric polynomials). The group of automorphisms of the F-manifold A_1^n which respect the Euler field E is the symmetric group S_n which permutes the coordinates u_1, \dots, u_n . The map $\Lambda^{(n)}$ is the quotient map for this group. It is branched along the bifurcation diagram $\mathcal{B} = \{u \mid u_i = u_j \text{ for some } i \neq j\}$. The image $\Lambda^{(n)}(\mathcal{B})$ is the hypersurface

$$\mathcal{D}^{(n)} := \{a \in \mathbb{C}^n \mid z^n + \sum a_i z^{n-i} \text{ has multiple roots}\} \subset \mathbb{C}^n. \quad (10.2)$$

The restriction $\Lambda^{(n)} : \mathbb{C}^n - \mathcal{B} \rightarrow \mathbb{C}^n - \mathcal{D}^{(n)}$ induces an F-manifold structure on $\mathbb{C}^n - \mathcal{D}^{(n)}$, with unit field

$$e^{(n)} := d\Lambda^{(n)}(e) = -n \frac{\partial}{\partial a_1} - \sum_{i \geq 2} (n - i + 1) a_{i-1} \frac{\partial}{\partial a_i} \quad (10.3)$$

and Euler field

$$E^{(n)} := d\Lambda^{(n)}(E) = \sum_i ia_i \frac{\partial}{\partial a_i} . \quad (10.4)$$

This F-manifold $(\mathbb{C}^n - \mathcal{D}^{(n)}, \circ, e^{(n)})$ will be denoted by A_1^n/S_n .

Theorem 10.2. *Let (M, \circ, e) be a massive F-manifold with generating function $F : L \rightarrow \mathbb{C}$ and Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$.*

Then $\Lambda^{-1}(\mathcal{D}^{(n)}) = \mathcal{B}$ and $d\Lambda(e) = e^{(n)}$. The restriction $\Lambda : M - \mathcal{B} \rightarrow \mathbb{C}^n - \mathcal{D}^{(n)}$ is an immersion and locally an isomorphism of F-manifolds. It maps the Euler field $\mathbf{a}^{-1}(F|_{M-\mathcal{B}})$ on $M - \mathcal{B}$ to the Euler field $E^{(n)}$.

Proof: In $M - \mathcal{K}$ the multiplication is semisimple and locally the values of the generating function are canonical coordinates. The map Λ factors on $M - \mathcal{B}$ locally into an isomorphism to A_1^n and into the map $\Lambda^{(n)}$. \square

The most important part of Theorem 10.2 is that $\Lambda : M - \mathcal{B} \rightarrow \mathbb{C}^n - \mathcal{D}^{(n)}$ is locally biholomorphic. The following statements for germs will also be useful.

Lemma 10.3. *Let $(M, p) = \prod_{k=1}^l (M_k, p)$ be a germ of a massive F-manifold with analytic spectrum L and with decomposition into irreducible germs (M_k, p) of dimension n_k , $\sum n_k = n$.*

a) *There exists precisely one generating function on the multigerms $(L, \pi^{-1}(p))$ for any choice of its values on $\pi^{-1}(p) = \{\lambda_1, \dots, \lambda_l\}$.*

b) *Choose a function $t_1 : (M, p) \rightarrow \mathbb{C}$ with $e(t_1) = 1$.*

The values of a generating function for the points in L above an orbit of e are of the form $t_1 + \text{constant}$.

The entry Λ_i of a Lyashko-Looijenga map $\Lambda : (M, p) \rightarrow \mathbb{C}^n$ is a polynomial of degree i in t_1 with coefficients in $\{g \in \mathcal{O}_{M,p} \mid e(g) = 0\}$ and leading coefficient $(-1)^i \binom{n}{i}$.

c) *Choose representatives M_k for the germs (M_k, p) and Lyashko-Looijenga maps $\Lambda^{[k]} : M_k \rightarrow \mathbb{C}^{n_k}$. Then the function $\Lambda = (\Lambda_1, \dots, \Lambda_n) : \prod_k M_k \rightarrow \mathbb{C}^n$ which is defined by*

$$z^n + \sum_{i=1}^n \Lambda_i z^{n-i} = \prod_{k=1}^l (z^{n_k} + \sum_{i=1}^{n_k} \Lambda_i^{[k]} z^{n_k-i}) \quad (10.5)$$

is a Lyashko-Looijenga map for the representative $\prod M_k$ of the germ (M, p) . Any Lyashko-Looijenga map for (M, p) is of this type.

Proof: a) Lemma 7.1.

- b) It suffices to prove the first part for an orbit of e in $M - \mathcal{K}$. There the generating function comes from an Euler field. (3.5) and (3.6) imply $\text{Lie}_e(E \circ) = \text{id}$. The values of F are the eigenvalues of $E \circ$.
- c) $\Lambda^{[k]}$ corresponds to an Euler field $E^{[k]}$ (at least) on $M_k - \mathcal{B}_{M_k}$. The sum $\sum_k E^{[k]}$ is an Euler field on $\prod (M_k - \mathcal{B}_{M_k})$ by Proposition 4.1. The corresponding generating function extends to $\prod M_k$ and has the given Λ as Lyashko-Looijenga map. The last statement follows with a). \square

Consider the projection $pr_e : (M, p) \rightarrow (M^{(r)}, p^{(r)})$ whose fibers are the orbits of e (chapter 9). The e -invariant hypersurfaces \mathcal{B} and \mathcal{K} project to hypersurfaces in $M^{(r)}$, which are called the restricted bifurcation diagram $\mathcal{B}^{(r)}$ and the restricted caustic $\mathcal{K}^{(r)}$. In chapter 8 the restricted Lagrange map was defined as the germ at $p^{(r)} \in M^{(r)}$ of a Lagrange map $L^{(r)} \hookrightarrow T^*M^{(r)} \rightarrow M^{(r)}$. Because of Corollary 8.3 the notion of a reduced Lyashko-Looijenga map is welldefined for the restricted Lagrange map (independently of the identification of the Lagrange fibration with the cotangent bundle of $M^{(r)}$).

The space of orbits of the field $e^{(n)}$ (formula (10.3)) in \mathbb{C}^n can be identified with $\{a \in \mathbb{C}^n \mid a_1 = 0\} = \{0\} \times \mathbb{C}^{n-1} \cong \mathbb{C}^{n-1}$ and is equipped with the coordinate system $(a_2, \dots, a_n) = a'$. The projection to this orbit space is denoted by $pr^{(n)} : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$, the image of $\mathcal{D}^{(n)}$ is

$$\mathcal{D}(A_{n-1}) := \{a' \in \mathbb{C}^{n-1} \mid z^n + \sum_{i=2}^n a_i z^{n-i} \text{ has multiple roots}\} \quad (10.6)$$

(it is isomorphic to the discriminant of the singularity or F-manifold A_{n-1} , cf. chapter 16).

Corollary 10.4. *Let (M, p) be the germ of a massive F-manifold, F a generating function, Λ ($\Lambda^{(red)}$) the (reduced) Lyashko-Looijenga map of (M, \circ, e, F) .*

$\Lambda^{(red)} : (M, p) \rightarrow \mathbb{C}^{n-1}$ is constant along the orbits of e . The induced map $\Lambda^{(red)(r)} : (M^{(r)}, p^{(r)}) \rightarrow \mathbb{C}^{n-1}$ is a reduced Lyashko-Looijenga map for the restricted Lagrange map. The following diagram commutes, the diagonal morphism is $\Lambda^{(red)} = pr^{(n)} \circ \Lambda = \Lambda^{(red)(r)} \circ pr_e$,

$$\begin{array}{ccc} (M, p) & \xrightarrow{\Lambda} & \mathbb{C}^n \\ \downarrow pr_e & & \downarrow pr^{(n)} \\ (M^{(r)}, p^{(r)}) & \xrightarrow{\Lambda^{(red)(r)}} & \mathbb{C}^{n-1} \end{array} \quad (10.7)$$

The restriction

$$\Lambda^{(red)(r)} : M^{(r)} - \mathcal{B}^{(r)} \rightarrow \mathbb{C}^{n-1} - \mathcal{D}(A_{n-1}) \quad (10.8)$$

is locally biholomorphic.

Proof: $\Lambda^{(red)}$ is constant along the orbits of e because of Lemma 10.3 b). The formulas (9.4), (9.5), (9.6) show that $\Lambda^{(red)(r)}$ is a reduced Lyashko-Looijenga map for the restricted Lagrange map. The rest follows from Theorem 10.2. \square

11. DISCRIMINANT OF AN F-MANIFOLD

Let (M, \circ, e, F) be a massive n -dimensional F-manifold with a generating function $F : L \rightarrow \mathbb{C}$ and Lyashko-Looijenga map $\Lambda = (\Lambda_1, \dots, \Lambda_n) : M \rightarrow \mathbb{C}^n$; the discriminant of (M, \circ, e, F) is the hypersurface $\mathcal{D} = \Lambda_n^{-1}(0) \subset M$ (Definition 10.1).

If F is holomorphic and $E = \mathbf{a}^{-1}(F)$ is its Euler field then $\Lambda_n = (-1)^n \cdot \det(E \circ)$, and the discriminant is the set of points where the multiplication with E is not invertible.

By definition of Λ_n , the discriminant is $\mathcal{D} = \pi(F^{-1}(0))$. Theorem 11.1 will give an isomorphism between $F^{-1}(0)$ and the development $\tilde{\mathcal{D}} \subset \mathbb{P}T^*M$ of \mathcal{D} .

We need an identification of subsets of T^*M and $\mathbb{P}T^*M$. The fiber-wise linear function on T^*M which corresponds to e is called y_1 . The canonical map

$$y_1^{-1}(1) \longrightarrow \mathbb{P}T^*M \quad (11.1)$$

identifies $y_1^{-1}(1) \subset T^*M$ with the open subset in $\mathbb{P}T^*M$ of hypersurfaces in the tangent spaces of M which do not contain $\mathbb{C} \cdot e$.

The restriction to $y_1^{-1}(1)$ of the canonical 1-form α on T^*M gives the contact structure on $y_1^{-1}(1)$ which is induced by the canonical contact structure on $\mathbb{P}T^*M$.

Theorem 11.1. *Let (M, \circ, e, F) and $F^{-1}(0) \subset L \subset y_1^{-1}(1) \subset T^*M$ be as above.*

*The canonical map $y_1^{-1}(1) \hookrightarrow \mathbb{P}T^*M$ identifies $F^{-1}(0)$ with the development $\tilde{\mathcal{D}} \subset \mathbb{P}T^*M$ of the discriminant \mathcal{D} .*

Proof: We want to make use of the discussion of fronts and graphs (chapter 8) and of the restricted Lagrange map (chapter 9). It is sufficient to consider the germ (M, p) for some $p \in \mathcal{D}$. We choose coordinates (t_1, \dots, t_n) and (y_1, \dots, y_n) as in the proof of Theorem 9.4. The generating function F for $L \hookrightarrow T^*M \rightarrow M$ takes the form

$$F(y', t) = t_1 + F^{(r)}(y', t') , \quad (11.2)$$

$F^{(r)}$ is a generating function of $L^{(r)}$ with respect to (9.6), (9.5), and $\alpha' = \sum_{i \geq 2} y_i dt_i$. The isomorphism with a sign

$$(-t_1, pr_e) : (M, p) \rightarrow (\mathbb{C} \times M^{(r)}, 0 \times p^{(r)}) \quad (11.3)$$

maps the discriminant to the front $\text{Im}(F^{(r)}, \pi^{(r)})$ of $L^{(r)}$ and $F^{(r)}$.

The development of this front is identified with the graph $\text{Im}(F^{(r)}, id) \subset \mathbb{C} \times T^*M^{(r)}$ of $F^{(r)} : L^{(r)} \rightarrow \mathbb{C}$, by Proposition 8.1 and the embedding (cf. formula (8.1))

$$\begin{aligned} \mathbb{C} \times T^*M^{(r)} &\hookrightarrow \mathbb{P}T^*(\mathbb{C} \times M^{(r)}), \\ (-t_1, \lambda) &\mapsto ((dt_1 + \lambda)^{-1}(0), (-t_1, \pi^{(r)}(\lambda))) . \end{aligned} \quad (11.4)$$

But (11.1), (11.3), and (11.4) together yield also an isomorphism $y_1^{-1}(1) \rightarrow \mathbb{C} \times T^*M^{(r)}$, which maps $F^{-1}(0)$ to this graph. \square

Remarks 11.2. i) In the proof only the choice of t_1 is essential. It is equivalent to various other choices: the choice of a section of the projection $(M, p) \rightarrow (M^{(r)}, p^{(r)})$, the choice of a 1-form α' for the Lagrange fibration in the restricted Lagrange map.

ii) Fixing such a choice of t_1 , one obtains together with α' and $F^{(r)}$ a Lyashko-Looijenga map $\Lambda^{(r)} : (M^{(r)}, p^{(r)}) \rightarrow \mathbb{C}^{n-1}$ for $F^{(r)}$. Then (11.3) identifies the entry Λ_n of the Lyashko-Looijenga map $\Lambda : (M, p) \rightarrow \mathbb{C}^n$ with the polynomial $(-t_1)^n + \sum_{i=2}^n \Lambda_i^{(r)} (-t_1)^{n-i}$.

iii) $F^{-1}(0) \subset L$ is not an analytic hypersurface of L at points of L_{sing} where F is not holomorphic. But Theorem 11.1 shows that it is everywhere a subvariety of L of pure codimension 1. Examples with not holomorphic F will be given in chapter 20.

iv) Let $(M, p) = \prod_{k=1}^l (M_k, p)$ be the decomposition of a germ (M, p) into irreducible germs of F-manifolds. A Lyashko-Looijenga map $\Lambda : (M, p) \rightarrow \mathbb{C}^n$ corresponds to Lyashko-Looijenga maps $\Lambda^{[k]} : (M_k, p) \rightarrow \mathbb{C}^{n_k}$ for the irreducible germs, in a way which was described in Lemma 10.3 c). Especially $\Lambda_n = \prod_k \Lambda_{n_k}^{[k]}$, and the germ (\mathcal{D}, p) of the discriminant is

$$(\mathcal{D}, p) = \bigcup_k \left(\prod_{j < k} M_j \times (\Lambda_{n_k}^{[k]})^{-1}(0) \times \prod_{j > k} M_j, p \right) , \quad (11.5)$$

the union of products of smooth germs with the discriminants for the irreducible germs (of course, $((\Lambda_{n_k}^{[k]})^{-1}(0), p) = \emptyset$ for some or all k is possible).

v) The development $\tilde{\mathcal{D}}$ of the discriminant \mathcal{D} gives the tangent hyperplanes to \mathcal{D} . Let $(M, p) = \prod_{k=1}^l (M_k, p)$ be as in iv), and $\pi^{-1}(p) = \{\lambda_1, \dots, \lambda_l\}$.

Theorem 11.1 says that the tangent hyperplanes to (\mathcal{D}, p) are those hyperplanes $\lambda_k^{-1}(0) \subset T_p M$ for which $F(\lambda_k) = 0$.

Especially, if $l = 1$ and $F(\lambda_1) = 0$, then $\lambda_1^{-1}(0) \subset T_p M$ is the nilpotent subalgebra of $T_p M$ and the unique tangent hyperplane to (\mathcal{D}, p) . The general case fits with Lemma 1.1 iv) and (11.5).

vi) $\alpha|_{L_{reg}} = dF|_{L_{reg}}$ shows immediately that $F^{-1}(0) \subset y_1^{-1}(1)$ is a Legendre subvariety.

vii) If $\lambda \in F^{-1}(0) \cap T_p^* M$, there is a canonical projection from the Legendre germ $(F^{-1}(0), \lambda)$ to one component of the Lagrange multigerms $(L^{(r)}, \pi^{(r)-1}(p^{(r)}))$. It is a bijective morphism. It is an isomorphism iff F is holomorphic at (L, λ) (see Remark 8.2 iii)).

One can recover the multiplication on a massive n -dimensional F-manifold M from the unit field e and a discriminant \mathcal{D} if the orbits of e are sufficiently large. To make this precise, we introduce the following notion.

Definition 11.3. A massive F-manifold (M, \circ, e, F) with generating function F is in *standard form* if there exists globally a projection $pr_e : M \rightarrow M^{(r)}$ to a manifold $M^{(r)}$ such that

- α) the fibers are the orbits of e (and thus connected),
- β) they are with their affine linear structure isomorphic to open (connected) subset of \mathbb{C} ,
- γ) the projection $pr_e : \mathcal{D} \rightarrow M^{(r)}$ is a branched covering of degree n .

Remarks 11.4. i) If (M, \circ, e, F) is a massive F-manifold with generating function F and properties α) and β) then

$$\left(-\frac{1}{n}\Lambda_1, pr_e\right) : M \rightarrow \mathbb{C} \times M^{(r)} \quad (11.6)$$

is an embedding because of $e(-\frac{1}{n}\Lambda_1) = 1$ (Lemma 10.3 b)).

The F-manifold M can be extended uniquely to an F-manifold isomorphic to $\mathbb{C} \times M^{(r)}$. Also the generating function F can be extended. The discriminant of this extended F-manifold satisfies γ) because of $\mathcal{D} = \Lambda_n^{-1}(0)$ and Lemma 10.3 b).

ii) For (M, \circ, e, F) as in i) the coordinate $t_1 := -\frac{1}{n}\Lambda_1$ is distinguished and, up to addition of a constant, even independent of the choice of F . Nevertheless it does not seem to have beautiful properties: In the case of the simplest 3-dimensional irreducible germs of F-manifolds, A_3 , B_3 , H_3 , it is not part of the coordinate system of a nice normal form (chapter 18). Using the data in [Du1] one can also check that $-\frac{1}{n}\Lambda_1$ is not a flat coordinate of the Frobenius manifolds A_3 , B_3 , H_3 .

Corollary 11.5. *Let (M, \circ, e, F) be a massive F -manifold with generating function F and in standard form.*

a) *The branch locus of the branched covering $pr_e : \mathcal{D} \rightarrow M^{(r)}$ is \mathcal{D}_{sing} , the set $pr_e(\mathcal{D}_{sing})$ of critical values is the restricted bifurcation diagram $\mathcal{B}^{(r)} = pr_e(\mathcal{B})$.*

b) *The union of the shifts of $F^{-1}(0)$ with the Hamilton field $\tilde{e} = H_{y_1}$ is the analytic spectrum $L \subset T^*M$.*

c) *The data (M, \circ, e, F) and (M, e, \mathcal{D}) are equivalent.*

Proof: a) Theorem 11.1 implies that all tangent hyperplanes to \mathcal{D} are transversal to the unit field. Therefore the branch locus is only \mathcal{D}_{sing} .

b)+c) \mathcal{D} and e determine $F^{-1}(0) \subset y_1^{-1}(1)$ because of Theorem 11.1. The union of shifts of $F^{-1}(0)$ with the Hamilton field $\tilde{e} = H_{y_1}$ is finite of degree n over M because of γ) and it is contained in L , so it is L . The function $F : L \rightarrow \mathbb{C}$ is determined by $F^{-1}(0) \subset L$ and by the linearity of F along the orbits of \tilde{e} . \square

Theorem 11.1 and Corollary 11.5 b) give one possibility to recover the multiplication of a massive F -manifold (M, \circ, e, F) in standard form from the discriminant \mathcal{D} and the unit field e . A more elementary way is the following.

Corollary 11.6. *Let (M, \circ, e, F) be a massive n -dimensional F -manifold with generating function F and in standard form. The multiplication can be recovered from the discriminant \mathcal{D} and the unit field e in the following way. The multiplication is semisimple outside of the bifurcation diagram $\mathcal{B} = pr_e^{-1}(pr_e(\mathcal{D}_{sing}))$. For a point $p \in M - \mathcal{B}$, the idempotent vectors $e_i(p) \in T_p M$ with $e_i(p) \circ e_j(p) = \delta_{ij} e_i(p)$ are uniquely determined by i) and ii):*

$$i) \ e(p) = \sum_{i=1}^n e_i(p),$$

ii) *the multigerms $(\mathcal{D}, \mathcal{D} \cap pr_e^{-1}(pr_e(p)))$ has exactly n tangent hyperplanes; their shifts to $T_p M$ with e are the hyperplanes $\bigoplus_{i \neq k} \mathbb{C} \cdot e_i \subset T_p M$, $k = 1, \dots, n$.*

Proof: Remark 11.2 v). \square

12. 2-DIMENSIONAL F -MANIFOLDS

The only 1-dimensional germ of an F -manifold is A_1 (Example 4.3 i)). The class of 3-dimensional germs of massive F -manifolds is already vast. Examples and a partial classification will be given in chapter 20. But the classification of 2-dimensional germs of F -manifolds is nice.

Theorem 12.1. *a) The only germs of 2-dimensional massive F-manifolds are, up to isomorphism, the germs $I_2(m)$, $m \in \mathbb{N}_{\geq 2}$, with $I_2(2) = A_1^2$, $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(5) =: H_2$, $I_2(6) = G_2$, from Example 4.3 iv):*

The multiplication on $(M, p) = (\mathbb{C}^2, 0)$ with coordinates t_1, t_2 and $\delta_i := \frac{\partial}{\partial t_i}$ is given by $e := \delta_1$ and $\delta_2 \circ \delta_2 = t_2^{m-2} \cdot \delta_1$. An Euler field of weight 1 is $E = t_1\delta_1 + \frac{2}{m}t_2\delta_2$. Its discriminant is $\mathcal{D} = \{t \mid t_2^2 - \frac{4}{m^2}t_2^m = 0\}$. $I_2(m)$ is irreducible for $m \geq 3$ with caustic and bifurcation diagram $\mathcal{K} = \mathcal{B} = \{t \mid t_2 = 0\}$. The space of Euler fields of weight d is $d \cdot E + \mathbb{C} \cdot e$ for $m \geq 3$.

b) The only germ of a 2-dimensional not massive F-manifold is the germ $(\mathbb{C}^2, 0)$ from Example 4.3 v) with multiplication given by $e := \delta_1$ and $\delta_2 \circ \delta_2 = 0$. The caustic is empty. An Euler field of weight 1 is $E = t_1\delta_1$. The space of all Euler fields of weight 0 is $\{\varepsilon_1\delta_1 + \varepsilon_2(t_2)\delta_2 \mid \varepsilon_1 \in \mathbb{C}, \varepsilon_2(t_2) \in \mathbb{C}\{t_2\}\}$.

Proof: a) Givental [Gi2](1.3, p 3253) classified the 1-dimensional miniversal germs of Lagrange maps with flat projection. Together with Theorem 9.4 this yields implicitly the classification of the 2-dimensional irreducible germs of massive F-manifolds. But we can recover this in a simple way and we need to be more explicit.

Let (M, p) be a 2-dimensional germ of a massive F-manifold with projection $pr_e : (M, p) \rightarrow (M^{(r)}, p^{(r)})$ to the space of orbits of e . There is a unique generating function $F : (L, \pi^{-1}(p)) \rightarrow (\mathbb{C}, 0)$. Its bifurcation diagram $\mathcal{B} \subset M$ and restricted bifurcation diagram $\mathcal{B}^{(r)} \subset M^{(r)}$ are the hypersurfaces $\mathcal{B} = pr_e^{-1}(p^{(r)})$ and $\mathcal{B}^{(r)} = \{p^{(r)}\}$.

By Corollary 10.4, the reduced Lyashko-Looijenga map $\Lambda^{(red)(r)} : (M^{(r)}, p^{(r)}) \rightarrow (\mathbb{C}, 0)$ of the restricted Lagrange map is a cyclic branched covering of some order \tilde{m} , and the Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^2$ is also a cyclic branched covering of order \tilde{m} , branched along \mathcal{B} .

Because of Corollary 8.3 and Theorem 9.4, this branching order \tilde{m} of $\Lambda^{(red)(r)}$ determines the germ (M, p) of the F-manifold up to isomorphism. It rests to determine the allowed \tilde{m} and explicit formulas for the F-manifolds.

Now consider the manifolds \mathbb{C}^2 with multiplication on $T\mathbb{C}^2$ given by $e = \delta_1$ and $\delta_2 \circ \delta_2 = t_2^{m-2} \cdot \delta_1$. The analytic spectrum $L \subset T^*\mathbb{C}^2$ is

$$L = \{(y_1, y_2, t_1, t_2) \mid y_1 = 1, y_2 \cdot y_2 = t_2^{m-2}\} . \quad (12.1)$$

It is an exercise to see that L is a Lagrange variety with generating function $F = t_1 + \frac{2}{m}y_2t_2$ with respect to $\alpha = y_1dt_1 + y_2dt_2$, i.e. one has $(\alpha - dF)|_{L_{reg}} = 0$. Then this gives an F-manifold and $E = \mathbf{a}^{-1}(F) = t_1\delta_1 + \frac{2}{m}t_2\delta_2$ is an Euler field of weight 1. The Lyashko-Looijenga map

Λ is

$$\Lambda : \mathbb{C}^2 \rightarrow \mathbb{C}^2, (t_1, t_2) \mapsto (-2t_1, t_1^2 - \frac{4}{m^2} t_2^m). \quad (12.2)$$

It is branched along $\mathcal{B} = \{t \mid t_2 = 0\}$ of degree m .

So $I_2(m)$ is the desired F-manifold for any branching order $m = \tilde{m} \geq 2$. The same calculation yields for $m = \tilde{m} = 1$ the F-manifold A_1^2/S_2 on $\mathbb{C}^2 - \{t \mid t_2 = 0\}$ (chapter 10) with meromorphic multiplication along $\{t \mid t_2 = 0\}$.

b) Let $(M, p) = (\mathbb{C}^2, 0)$ be the germ of a 2-dimensional not massive F-manifold with $e = \delta_1$. Then $\text{Lie}_e(\circ) = 0$ and $[e, \delta_2] = 0$ imply $\text{Lie}_e(\delta_2 \circ \delta_2) = 0$. Hence $\delta_2 \circ \delta_2 = \beta(t_2)\delta_1 + \gamma(t_2)\delta_2$ for some $\beta(t_2), \gamma(t_2) \in \mathbb{C}\{t_2\}$.

The field $\tilde{\delta}_2 := \delta_2 - \frac{1}{2}\gamma(t_2)\delta_1$ satisfies $\tilde{\delta}_2 \circ \tilde{\delta}_2 = (\beta(t_2) + \frac{1}{4}\gamma(t_2)^2)\delta_1$ and $[\delta_1, \tilde{\delta}_2] = 0$. Changing coordinates we may suppose $\delta_2 = \tilde{\delta}_2$, $\gamma(t_2) = 0$.

The analytic spectrum is $L = \{(y_1, y_2, t_1, t_2) \mid y_1 = 1, y_2 \cdot y_2 = \beta(t_2)\}$. The F-manifold is not massive, hence $\beta(t_2) = 0$. One checks with (3.3) and (3.4) easily that this multiplication gives an F-manifold and that the space of Euler fields is as claimed. \square

Let us discuss the role which 2-dimensional germs of F-manifolds can play for higher dimensional massive F-manifolds. The set

$$\mathcal{D}^{(n,3)} := \{a \in \mathbb{C}^n \mid z^n + \sum_{i=1}^n a_i z^{n-i} \text{ has a root} \quad (12.3)$$

$$\text{of multiplicity} \geq 3\} \subset \mathcal{D}^{(n)} \subset \mathbb{C}^n$$

is an algebraic subvariety of \mathbb{C}^n of codimension 2 (see the proof of Proposition 2.3). Given a massive F-manifold M , the space

$$\mathcal{K}^{(3)} := \{p \in M \mid \mathcal{P}(T_p M) \succ (3, 1, \dots, 1)\} \subset \mathcal{K} \subset M \quad (12.4)$$

of points p such that (M, p) does not decompose into 1- and 2-dimensional germs of F-manifolds is empty or an analytic subvariety (Proposition 2.3).

Theorem 12.2. *Let (M, \circ, e) be a massive F-manifold with generating function F .*

a) F is holomorphic on $\pi^{-1}(M - \mathcal{K}^{(3)})$ and gives rise to an Euler field of weight 1 on $M - \mathcal{K}^{(3)}$.

b) If $\text{codim } \mathcal{K}^{(3)} \geq 2$ then F is holomorphic on L and $E = \mathbf{a}^{-1}(F)$ is an Euler field of weight 1 on M .

c) $\mathcal{K}^{(3)} \subset \Lambda^{-1}(\mathcal{D}^{(n,3)})$, and $\overline{\Lambda^{-1}(\mathcal{D}^{(n,3)}) - \mathcal{K}^{(3)}}$ is analytic of pure codimension 2. Thus $\text{codim } \mathcal{K}^{(3)} \geq 2 \iff \text{codim } \Lambda^{-1}(\mathcal{D}^{(n,3)}) \geq 2$.

d) *The restriction of the Lyashko-Looijenga map*

$$\Lambda : M - \Lambda^{-1}(\mathcal{D}^{(n,3)}) \rightarrow \mathbb{C}^n - \mathcal{D}^{(n,3)}$$

is locally a branched covering, branched along $\mathcal{B} - \Lambda^{-1}(\mathcal{D}^{(n,3)})$.

If $p \in \mathcal{B} - \Lambda^{-1}(\mathcal{D}^{(n,3)})$ and $\Lambda(p) \in \mathcal{D}^{(n)} - \mathcal{D}^{(n,3)}$ are smooth points of the hypersurfaces \mathcal{B} and $\mathcal{D}^{(n)}$ and if there the branching order is m , then (M, p) is the germ of an F-manifold of type $I_2(m) \times A_1^{n-2}$.

Proof: a)+b) Each germ (L, λ_k) of the analytic spectrum $(L, \pi^{-1}(p)) = (L, \{\lambda_1, \dots, \lambda_l\})$ of a reducible germ $(M, p) = \prod_k (M_k, p)$ is the product of a smooth germ with the analytic spectrum of (M_k, p) . The analytic spectrum of $I_2(m)$ ($m \geq 2$) is isomorphic to $(\mathbb{C}, 0) \times (\{y_2, t_2\} \mid y_2^2 = t_2^{m-2}\}, 0)$. One applies Lemma 7.6.

c) A Lyashko-Looijenga map of $I_2(m)$ is a cyclic branched covering of order m , branched along the bifurcation diagram. This together with Lemma 10.3 c) implies that locally around a point $p \in M - \mathcal{K}^{(3)}$ the fibers of the Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$ are finite. Therefore $\text{codim}_M(\Lambda^{-1}(\mathcal{D}^{(n,3)}), p) = \text{codim}_{\mathbb{C}^n}(\mathcal{D}^{(n,3)}) = 2$.

d) Λ determines the multiplication of the F-manifold M (Theorem 10.2). One uses this, Lemma 10.3 c) and properties of $I_2(m)$. \square

Many interesting F-manifolds, e.g. those for hypersurface singularities, boundary singularities, finite Coxeter groups (chapters 16, 17, 18), satisfy the property $\text{codim } \mathcal{K}^{(3)} \geq 2$ and have an Euler field of weight 1.

13. LOGARITHMIC VECTOR FIELDS

K. Saito [SaK3] introduced the notions of logarithmic vector fields and free divisors. Let $H \subset M$ be a reduced hypersurface in an n -dimensional manifold M . The sheaf $\text{Der}_M(\log H) \subset \mathcal{T}_M$ of logarithmic vector fields consists of those holomorphic vector fields which are tangent to H_{reg} . The hypersurface H is a free divisor if $\text{Der}_M(\log H)$ is a free \mathcal{O}_M -module of rank n .

The results in this chapter are not really new. They had been established in various generality by Bruce [Bru], Givental [Gi2](chapter 1.4), Lyashko [Ly1][Ly3], K. Saito [SaK4][SaK5], Terao [Ter], and Zakalyukin [Za] as results for hypersurface singularities, boundary singularities or miniversal Lagrange maps. But the formulation using the multiplication of F-manifolds is especially nice.

Theorem 13.1. *Let (M, \circ, e) be a massive F-manifold with Euler field E of weight 1, generating function $F = \mathbf{a}(E)$ and discriminant $\mathcal{D} = (\det(E \circ))^{-1}(0) = \pi(F^{-1}(0))$.*

- a) The discriminant is a free divisor with $\text{Der}_M(\log \mathcal{D}) = E \circ \mathcal{T}_M$.
 b) The kernel of the map

$$\mathbf{a}_{\mathcal{D}} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_{F^{-1}(0)}, \quad X \mapsto \mathbf{a}(X)|_{F^{-1}(0)} \quad (13.1)$$

is

$$\ker \mathbf{a}_{\mathcal{D}} = E \circ \mathcal{T}_M = \text{Der}_M(\log \mathcal{D}) . \quad (13.2)$$

Proof: a) $E \circ \mathcal{T}_M$ is a free \mathcal{O}_M -module of rank n . Therefore a) follows from (13.2).

b) The \mathcal{O}_M -module $\pi_* \mathcal{O}_{F^{-1}(0)}$ has support \mathcal{D} . (13.2) holds in $M - \mathcal{D}$. The set $\mathcal{D} \cap \mathcal{B}$ has codimension 2 in M . Hence (13.2) holds in M if it holds in $\mathcal{D} - \mathcal{B}$.

Let $p \in \mathcal{D} - \mathcal{B}$. We choose a small neighborhood U of p with canonical coordinates u_1, \dots, u_n centered at p , with $\mathcal{D} \cap U = \{u \mid u_1 = 0\}$ and with Euler field $E = u_1 e_1 + \sum_{i \geq 2} (u_i + r_i) e_i$ for some $r_i \in \mathbb{C} - \{0\}$. With the notation of the proof of Theorem 6.2 ii) \Rightarrow iii) we have $\alpha = \sum x_i du_i$,

$$F^{-1}(0) \cap \pi^{-1}(U) = \{(x, u) \mid x_j = \delta_{1j}, u_1 = 0\} ,$$

and for any vector field $X = \sum \xi_i e_i \in \mathcal{T}_M(U)$

$$\mathbf{a}(X)|_{F^{-1}(0) \cap \pi^{-1}(U)} = \xi_1(0, u_2, \dots, u_n) .$$

Therefore

$$\begin{aligned} (\ker \mathbf{a}_{\mathcal{D}})_p &= \mathcal{O}_{M,p} \cdot u_1 e_1 \oplus \bigoplus_{i=2}^n \mathcal{O}_{M,p} \cdot e_i \\ &= E \circ \mathcal{T}_{M,p} = \text{Der}_{M,p}(\log \mathcal{D}) . \end{aligned} \quad (13.3)$$

□

Remark 13.2. One can see 13.1 a) in a different way: there is a criterion of K. Saito [SaK3](Lemma (1.9)). To apply it, one has to show

$$[E \circ \mathcal{T}_M, E \circ \mathcal{T}_M] \subset E \circ \mathcal{T}_M . \quad (13.4)$$

With (3.1) and (3.2) one calculates for any two (local) vector fields X, Y

$$[E \circ X, E \circ Y] = E \circ ([X, E \circ Y] - [Y, E \circ X] - E \circ [X, Y]) . \quad (13.5)$$

In the rest of this chapter (M, \circ, e, E) will be a massive F-manifold which is equipped with an Euler field E of weight 1 and which is in standard form (Definition 11.3). $pr_e : M \rightarrow M^{(r)}$ is the projection to the space of orbits of e . The sheaf of e -invariant vector fields

$$\mathcal{G} := \{X \in (pr_e)_* \mathcal{T}_M \mid [e, X] = 0\} \quad (13.6)$$

is a free $\mathcal{O}_{M^{(r)}}$ -module of rank n . Because of $\text{Lie}_e(\circ) = 0$ it is also an $\mathcal{O}_{M^{(r)}}$ -algebra.

Theorem 13.3. *Let (M, \circ, e, E) be a massive F -manifold with Euler field E of weight 1 and in standard form.*

a)

$$(pr_e)_* \mathcal{T}_M = \mathcal{G} \oplus (pr_e)_*(E \circ \mathcal{T}_M) . \quad (13.7)$$

b) *The kernel of the map*

$$\begin{aligned} (pr_e)_* \mathbf{a}_{\mathcal{D}} : (pr_e)_* \mathcal{T}_M &\rightarrow (pr_e \circ \pi)_* \mathcal{O}_{F^{-1}(0)}, \\ X &\mapsto \mathbf{a}(X)|_{F^{-1}(0)} \end{aligned} \quad (13.8)$$

is $(pr_e)_(E \circ \mathcal{T}_M)$. The restriction*

$$(pr_e)_* \mathbf{a}_{\mathcal{D}} : \mathcal{G} \rightarrow (pr_e \circ \pi)_* \mathcal{O}_{F^{-1}(0)} \quad (13.9)$$

is an isomorphism of $\mathcal{O}_{M^{(r)}}$ -algebras.

Proof: a) follows from b).

b) The kernel of $(pr_e)_* \mathbf{a}_{\mathcal{D}}$ is $(pr_e)_*(E \circ \mathcal{T}_M)$ because of Theorem 13.1 b). The F -manifold in standard form has a global restricted Lagrange map $L^{(r)} \hookrightarrow T^*M^{(r)} \rightarrow M^{(r)}$ (the identification of its Lagrange fibration with $T^*M^{(r)} \rightarrow M^{(r)}$ is unique only up to shifts in the fibers).

The canonical projection $F^{-1}(0) \rightarrow L^{(r)}$ is bijective (Corollary 11.5 b)), and then an isomorphism because F is holomorphic. It induces an isomorphism $(pr_e \circ \pi)_* \mathcal{O}_{F^{-1}(0)} \cong (\pi^{(r)})_* \mathcal{O}_{L^{(r)}}$. The composition with (13.9) is the isomorphism

$$\mathbf{a}^{(r)} : \mathcal{G} \rightarrow (\pi^{(r)})_* \mathcal{O}_{L^{(r)}} \quad (13.10)$$

from Lemma 9.5. \square

Theorem 13.1 and Theorem 13.3 are translations to F -manifolds of statements in [SaK4](1.6)[SaK5](1.7) for hypersurface singularities. In fact, K. Saito used essentially (13.9) to define the multiplication on \mathcal{G} for hypersurface singularities.

The arguments in Lemma 13.4 and Theorem 13.5 are due to Lyashko [Ly1][Ly3] and Terao [Ter], see also Bruce [Bru].

Again (M, \circ, e, E) is a massive F -manifold with Euler field of weight 1 and in standard form. We choose a function $t_1 : M \rightarrow \mathbb{C}$ with $e(t_1) = 1$ (e.g. $t_1 = -\frac{1}{n}\Lambda_1$, cf. Lemma 10.3 b)). This choice simplifies the formulation of the results in Lemma 13.4. The vector fields in $\mathcal{T}_{M^{(r)}}$ will be identified with their (unique) lifts in $\{X \in \mathcal{G} \mid X(t_1) = 0\} \subset \mathcal{G} \subset (pr_e)_* \mathcal{T}_M$. The projection to $\mathcal{T}_{M^{(r)}}$ of all possible lifts to M of vector fields in $M^{(r)}$ is

$$d(pr_e) : \mathcal{T}_{M^{(r)}} \oplus (pr_e)_* \mathcal{O}_M \cdot e \rightarrow \mathcal{T}_{M^{(r)}} . \quad (13.11)$$

Lemma 13.4. *Let (M, \circ, e, E, t_1) be as above.*

a)

$$\begin{aligned} & (pr_e)_*(E \circ \mathcal{T}_M) \cap \left(\mathcal{T}_{M^{(r)}} \oplus \bigoplus_{k=0}^{n-1} \mathcal{O}_{M^{(r)}} \cdot t_1^k \cdot e \right) \\ &= \bigoplus_{k=1}^{n-1} \mathcal{O}_{M^{(r)}} \cdot (t_1^k \cdot e - (t_1 e - E)^{\circ k}) . \end{aligned} \quad (13.12)$$

b) *Each vector field in $M^{(r)}$ which lifts to a vector field in $(pr_e)_* \text{Der}_M(\log \mathcal{D})$ lifts to a unique vector field in (13.12).*

c) *The vector fields in $M^{(r)}$ which lift to vector fields in $(pr_e)_* \text{Der}_M(\log \mathcal{D})$ are tangent to the restricted bifurcation diagram $\mathcal{B}^{(r)} = pr_e(\mathcal{B}) \subset M^{(r)}$ and form the free $\mathcal{O}_{M^{(r)}}$ -module of rank $n-1$*

$$\bigoplus_{k=1}^{n-1} \mathcal{O}_{M^{(r)}} \cdot d(pr_e)((t_1 e - E)^{\circ k}) \subset \text{Der}_{M^{(r)}}(\log \mathcal{B}^{(r)}) . \quad (13.13)$$

Proof: a) One sees inductively by multiplication with $t_1 e - E$ that for any $k \geq 1$

$$t_1^k e - (t_1 e - E)^{\circ k} \in (pr_e)_*(E \circ \mathcal{T}_M) \quad (13.14)$$

holds. $t_1 e - E \in \mathcal{G}$ and $\text{Lie}_e(\circ) = 0$ imply $(t_1 e - E)^{\circ k} \in \mathcal{G}$, therefore

$$\mathcal{T}_{M^{(r)}} \oplus \bigoplus_{k=0}^{n-1} \mathcal{O}_{M^{(r)}} t_1^k e = \mathcal{G} \oplus \bigoplus_{k=1}^{n-1} \mathcal{O}_{M^{(r)}} (t_1^k e - (t_1 e - E)^{\circ k}) . \quad (13.15)$$

Now the decomposition (13.7) yields (13.12).

b) $pr_e : \mathcal{D} \rightarrow M^{(r)}$ is a branched covering of degree n (Definition 11.3), so

$$\bigoplus_{k=0}^{n-1} \mathcal{O}_{M^{(r)}} \cdot t_1^k \rightarrow (pr_e)_*(\mathcal{O}_{\mathcal{D}}) \quad (13.16)$$

is an isomorphism. Therefore any lift $h \cdot e + X$, $h \in (pr_e)_* \mathcal{O}_M$, of $X \in \mathcal{T}_{M^{(r)}}$ can be replaced by a unique lift $\tilde{h} \cdot e + X$ with $\tilde{h} \in \bigoplus_{k=0}^{n-1} \mathcal{O}_{M^{(r)}} t_1^k$ and $(h - \tilde{h})|_{\mathcal{D}} = 0$. If $h \cdot e + X$ is tangent to \mathcal{D} , then $(h - \tilde{h})|_{\mathcal{D}} = 0$ is necessary and sufficient for $\tilde{h} \cdot e + X$ to be tangent to \mathcal{D} .

c) A generic point $p^{(r)} \in (\mathcal{B}^{(r)})_{reg}$ has a preimage $p \in (\mathcal{D}_{sing})_{reg}$ such that the projection of germs $pr_e : (\mathcal{D}_{sing}, p) \rightarrow (\mathcal{B}^{(r)}, p^{(r)})$ is an isomorphism (Corollary 11.5 a)). A vector field $h \cdot e + X$, $X \in \mathcal{T}_{M^{(r)}}$, which is tangent to \mathcal{D}_{reg} is also tangent to $(\mathcal{D}_{sing})_{reg}$. Then X is tangent to $(\mathcal{B}^{(r)})_{reg}$. One obtains the generators in (13.13) by projection to $\mathcal{T}_{M^{(r)}}$ of the generators in (13.12). \square

$\mathcal{K}^{(3)} \subset \mathcal{K} \subset M$ is the set of points $p \in M$ such that (M, p) does not decompose into 1- and 2-dimensional germs of F-manifolds (chapter 12).

Theorem 13.5. *Let (M, \circ, e, E) be a massive F-manifold with Euler field E of weight 1 and in standard form. Suppose that $\text{codim } \mathcal{K}^{(3)} \geq 2$.*

Then the restricted bifurcation diagram $\mathcal{B}^{(r)}$ is a free divisor and (13.13) is an equality.

Proof: In view of Lemma 13.4 c) it is sufficient to show that any vector field tangent to $\mathcal{B}^{(r)}$ lifts to a vector field tangent to \mathcal{D} .

$pr_e : \mathcal{D} - \mathcal{B} \rightarrow M^{(r)} - \mathcal{B}^{(r)}$ is a covering of degree n . For any vector field $X \in \mathcal{T}_{M^{(r)}}$ there exists a unique function $h_X \in (pr_e)_* \mathcal{O}_{\mathcal{D}-\mathcal{B}}$ such that $\tilde{h} \cdot e + X$ is tangent to $\mathcal{D} - \mathcal{B}$ iff $\tilde{h}|_{\mathcal{D}-\mathcal{B}} = h_X$.

One has to show that h_X extends to a function in $(pr_e)_* \mathcal{O}_{\mathcal{D}}$ if $X \in \text{Der}_{M^{(r)}}(\log \mathcal{B}^{(r)})$. Then the unique lift $\tilde{h} \cdot e + X$ with $\tilde{h} \in \bigoplus_{k=0}^{n-1} \mathcal{O}_{M^{(r)}} t_1^k$ and $\tilde{h}|_{\mathcal{D}} = h_X$ is tangent to \mathcal{D} .

Let p be a point in the set

$$\{p \in (\mathcal{D}_{\text{sing}})_{\text{reg}} \mid p^{(r)} \in (\mathcal{B}^{(r)})_{\text{reg}}, \\ pr_e : (\mathcal{D}, p) \rightarrow (M^{(r)}, p^{(r)}) \text{ has degree } 2\} . \quad (13.17)$$

Then the germ (\mathcal{D}, p) is the product of $(\mathbb{C}^{n-2}, 0)$ and the discriminant of the germ of an F-manifold of type $I_2(m)$ ($m \geq 2$) (Remark 11.2 iv)). One can find coordinates $(t_1, t') = (t_1, \dots, t_n)$ around $p \in M$ such that $(\mathcal{D}, p) \subset (M, p) \rightarrow (M^{(r)}, p^{(r)})$ corresponds to

$$(\{(t_1, t') \mid t_1^2 - t_2^m = 0\}, 0) \subset (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0), \quad t \mapsto t' . \quad (13.18)$$

Then $(\mathcal{B}^{(r)}, p^{(r)}) \cong (\{t' \mid t_2 = 0\}, 0)$. Obviously the vector fields tangent to $(\mathcal{B}^{(r)}, p^{(r)})$ have locally lifts to vector fields tangent to (\mathcal{D}, p) . The function h_X of a field $X \in \text{Der}_{M^{(r)}}(\log \mathcal{B}^{(r)})$ extends holomorphically to the set in (13.17). The complement in \mathcal{D} of $\mathcal{D}_{\text{reg}} = \mathcal{D} - \mathcal{B}$ and of the set in (13.17) has codimension ≥ 2 because of $\text{codim } \mathcal{K}^{(3)} \geq 2$. Therefore $h_X \in (pr_e)_* \mathcal{O}_{\mathcal{D}}$. \square

14. ISOMORPHISMS AND MODALITY OF GERMS OF F-MANIFOLDS

The following three results are applications of Theorem 10.2 for the Lyashko-Looijenga map. They will be proved together. $((M, p), \circ, e, \Lambda)$ denotes the germ of an F-manifold with the function germ $\Lambda : (M, p) \rightarrow \mathbb{C}^n$ as additional structure. A map germ $\varphi : (M, p) \rightarrow (M, p)$ respects Λ if $\Lambda \circ \varphi = \Lambda$.

Theorem 14.1. *The automorphism group of a germ (M, p) of a massive F-manifold is finite.*

Theorem 14.2. *Let (M, \circ, e, F) be a massive F -manifold with generating function F and Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$. For any $p_1 \in M$ the set*

$$\{q \in M \mid ((M, p_1), \circ, e, \Lambda) \cong ((M, q), \circ, e, \Lambda)\}$$

is discret and closed in M .

Corollary 14.3. *Let (M, \circ, e, E) be a massive F -manifold with Euler field E of weight 1. For any $p_1 \in M$ the set*

$$\{q \in M \mid ((M, p_1), \circ, e, E) \cong ((M, q), \circ, e, E)\}$$

is discret and closed in M .

Proof: 14.3 follows from 14.2. For Theorem 14.1, it suffices to regard an irreducible germ of a massive F -manifold. The automorphisms of an irreducible germ respect a given Lyashko-Looijenga map Λ because of Lemma 10.3 a). So we may fix for 14.1 and 14.2 a massive F -manifold (M, \circ, e) and a Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$.

The set

$$\Delta := \{(p, p') \in M \times M \mid \Lambda(p) = \Lambda(p')\}$$

has a reduced complex structure. It is a subset of $(M - \mathcal{B}) \times (M - \mathcal{B}) \cup \mathcal{B} \times \mathcal{B}$ and the intersection $\Delta \cap (M - \mathcal{B}) \times (M - \mathcal{B})$ is smooth of dimension n . This follows from Theorem 10.2

Now consider an isomorphism $\varphi : ((M, p), \circ, e, \Lambda) \rightarrow ((M, p'), \circ, e, \Lambda)$. The graph germ

$$(\mathcal{G}(\varphi), (p, p')) := (\{(q, \varphi(q)) \in M \times M \mid q \text{ near } p\}, (p, p'))$$

is a smooth analytic germ of dimension n and is contained in the germ $(\Delta, (p, p'))$. It meets $\Delta \cap (M - \mathcal{B}) \times (M - \mathcal{B})$. Because of the purity of the dimension of an irreducible analytic germ, it is an irreducible component of the analytic germ $(\Delta, (p, p'))$. One can recover the map germ φ from the graph germ $(\mathcal{G}(\varphi), (p, p'))$. The germ $(\Delta, (p, p'))$ consists of finitely many irreducible components. The case $p = p'$ together with the remarks at the beginning of the proof gives Theorem 14.1.

For Theorem 14.2, we assume that there is an infinite sequence $(p_i, \varphi_i)_{i \in \mathbb{N}}$ of different points $p_i \in M$ and map germs

$$\varphi_i : ((M, p_1), \circ, e, \Lambda) \xrightarrow{\cong} ((M, p_i), \circ, e, \Lambda)$$

and one accumulation point $p_\infty \in M$. The set $\overline{\Delta - \mathcal{B} \times \mathcal{B}}$ is analytic of pure dimension n . It contains the germs $(\mathcal{G}(\varphi_i), (p_1, p_i))$ and the point (p_1, p_∞) .

We can choose a suitable open neighborhood U of (p_1, p_∞) in $M \times M$ and a stratification

$$\bigcup_{\alpha} S_{\alpha} = U \cap \overline{\Delta - \mathcal{B} \times \mathcal{B}}$$

of $U \cap \overline{\Delta - \mathcal{B} \times \mathcal{B}}$ which consists of finitely many disjoint smooth connected constructible sets S_{α} and satisfies the boundary condition: *The boundary $\overline{S_{\alpha}} - S_{\alpha}$ of a stratum S_{α} is a union of other strata.*

$(\mathcal{G}(\varphi_i), (p_1, p_i))$ is an n -dimensional irreducible component of the n -dimensional germ $(\overline{\Delta - \mathcal{B} \times \mathcal{B}}, (p_1, p_i))$. There is a unique n -dimensional stratum whose closure contains $(\mathcal{G}(\varphi_i), (p_1, p_i))$. If $(p_1, p_i) \in S_{\alpha}$ then this together with the boundary condition implies $(S_{\alpha}, (p_1, p_i)) \subset (\mathcal{G}(\varphi_i), (p_1, p_i))$. The germ $(\mathcal{G}(\varphi_i), (p_1, p_i))$ is the graph of the isomorphism φ_i . Therefore it intersects the germ $(\{p_1\} \times M, (p_1, p_i))$ only in (p_1, p_i) ; the same holds for $(S_{\alpha}, (p_1, p_i))$.

Now there exists at least one stratum S_{α_0} which contains infinitely many of the points (p_1, p_i) . The intersection of the analytic sets $\overline{S_{\alpha_0}}$ and $U \cap (\{p_1\} \times M)$ contains these points as isolated points. This is impossible. The assumption above was wrong. \square

In singularity theory there are the notions of μ -constant stratum and (proper) modality of an isolated hypersurface singularity. One can define versions of them for the germ (M, p) of an F-manifold (M, \circ, e) (massive or not massive): The μ -constant stratum (S_{μ}, p) is the analytic germ of points $q \in M$ such that the eigenspace decompositions of $T_q M$ and $T_p M$ have the same partition (cf. Proposition 2.3).

The idempotent fields e_1, \dots, e_l of the decomposition $(M, p) = \prod_{k=1}^l (M_k, p)$ into irreducible germs of F-manifolds commute and satisfy $\text{Lie}_{e_i}(\circ) = 0 \cdot \circ$. So the germs (M, q) of points q in one integral manifold of e_1, \dots, e_l are isomorphic. This motivates the definition of the *modality*:

$$\text{mod}_{\mu}(M, p) := \dim(S_{\mu}, p) - l . \quad (14.1)$$

Let $(S_{\mu}^{[k]}, p)$ denote the μ -constant stratum of (M_k, p) ; Then Theorem 4.2 implies

$$(S_{\mu}, p) = \prod_k (S_{\mu}^{[k]}, p) \quad \text{and} \quad (14.2)$$

$$\text{mod}(M, p) = \sum_k \text{mod}_{\mu}(M_k, p) . \quad (14.3)$$

For massive F-manifolds, Theorem 14.2 and Lemma 10.3 give more information:

Corollary 14.4. *Let $(M, p) = \prod_{k=1}^l (M_k, p)$ be the germ of a massive F -manifold and $\Lambda : (M, p) \rightarrow \mathbb{C}^n$ a Lyashko-Looijenga map.*

a) There exist a representative S_μ of the μ -constant stratum (S_μ, p) , a neighborhood $U \subset \mathbb{C}^l$ of 0 and an isomorphism

$$\psi : S_\mu \rightarrow (S_\mu \cap \Lambda^{-1}(\Lambda(p))) \times U \quad (14.4)$$

such that $\psi^{-1}(\{q\} \times U)$ is the integral manifold of e_1, \dots, e_l which contains q . Any subset of points in $S_\mu \cap \Lambda^{-1}(\Lambda(p))$ with isomorphic germs of F -manifolds is discret and closed.

b)

$$\text{mod}_\mu(M, p) = \dim(S_\mu \cap \Lambda^{-1}(\Lambda(p)), p) , \quad (14.5)$$

$$\sup(\text{mod}_\mu(M, q) \mid q \text{ near } p) = \dim(\Lambda^{-1}(\Lambda(p)), p) . \quad (14.6)$$

Proof: a) For $l = 1$, the existence of ψ follows from the e -invariance of S_μ and from $e(-\frac{1}{n}\Lambda_1) = 1$ (Lemma 10.3 b)).

For arbitrary l , one uses (14.2) and Lemma 10.3 c): the maps Λ and $(\Lambda^{[1]}, \dots, \Lambda^{[l]})$ have the same germs of fibers, especially

$$S_\mu \cap \Lambda^{-1}(\Lambda(p)) = \prod_k S_\mu^{[k]} \cap \Lambda^{[k]-1}(\Lambda^{[k]}(p)) . \quad (14.7)$$

A germ (M, q) has only a finite number of Lyashko-Looijenga maps with fixed value at q (Lemma 10.3 a)). The finiteness statement in 14.4 a) follows from this and Theorem 14.2.

b) (14.5) follows from a). A representative of the germ $(\Lambda^{-1}(\Lambda(p)), p)$ is stratified into constructible subsets which consist of the points q with the same partition for the eigenspace decomposition of $T_q M$ (Proposition 2.3). A point $q \in \Lambda^{-1}(\Lambda(p))$ in an open stratum with maximal dimension satisfies

$$\begin{aligned} \text{mod}_\mu(M, q) &= \dim(S_\mu(q) \cap \Lambda^{-1}(\Lambda(p)), q) \\ &= \dim(\Lambda^{-1}(\Lambda(p)), q) = \dim(\Lambda^{-1}(\Lambda(p)), p) . \end{aligned} \quad (14.8)$$

This shows

$$\sup(\text{mod}_\mu(M, q) \mid q \in \Lambda^{-1}(\Lambda(p)) \text{ near } p) = \dim(\Lambda^{-1}(\Lambda(p)), p) . \quad (14.9)$$

The upper semicontinuity of the fiber dimension of Λ gives (14.6). \square

Remark 14.5. Gabrielov [Ga] proved in the case of isolated hypersurface singularities the upper semicontinuity of the modality,

$$\text{mod}_\mu(M, q) \leq \text{mod}_\mu(M, p) \text{ for } q \text{ near } p \quad (14.10)$$

(and the equality with another version of modality which was defined by Arnold). He used (14.5), (14.6), and a result of himself, Lazzeri,

and $\text{L}\hat{\text{e}}$, which, translated to the F-manifold of a singularity (chapter 16), says:

$$(S_\mu \cap \Lambda^{-1}(\Lambda(p)), p) = (\Lambda^{-1}(\Lambda(p)), p) . \quad (14.11)$$

(14.10) is an immediate consequence of (14.5), (14.6) and (14.11). But for other F-manifolds (14.11) and (14.10) are not clear.

In the case of the simple hypersurface singularities, the base of the semiuniversal unfolding is an F-manifold $M \cong \mathbb{C}^n$ and the map $\Lambda : M - \mathcal{B} \rightarrow \mathbb{C}^n - \mathcal{D}^{(n)}$ is a finite covering. Therefore the complement $M - \mathcal{B}$ is a $K(\pi, 1)$ space and the fundamental group is a subgroup of finite index of the braid group $Br(n)$. This is the application of Looijenga [Lo1] and Lyashko [Ar1] of the map Λ , which led to the name Lyashko-Looijenga map.

It can be generalized to F-manifolds. We call a massive F-manifold M *simple* if $\text{mod}_\mu(M, p) = 0$ for all $p \in M$. This fits with the notions of simple hypersurface singularities, simple boundary singularities, and simple Lagrange maps ([Gi2] 1.3, p 3251).

A distinguished class of simple F-manifolds are the F-manifolds of the finite Coxeter groups (chapter 18 and [Lo1][Ar1][Ly1][Ly3][Gi2]). There are other examples (Proposition 20.6 and Remark 20.7).

A Lyashko-Looijenga map of a massive F-manifold is locally a branched covering iff M is simple ((14.6) and Theorem 10.2). A detailed proof of the following result had been given by Looijenga [Lo1] (Theorem 2.1) (cf. also [Gi2] 1.4, Theorem 5).

Theorem 14.6. *Let $(M, p) = (\mathbb{C}^n, p)$ be the germ of a simple F-manifold with fixed coordinates. Then, if $\varepsilon < \varepsilon_0$ for some ε_0 , the space $\{z \in \mathbb{C}^n \mid |z| < \varepsilon\} - \mathcal{B}$ is a $K(\pi, 1)$ space. Its fundamental group is a subgroup of finite index of the braid group $Br(n)$.*

15. ANALYTIC SPECTRUM EMBEDDED DIFFERENTLY

The analytic spectrum $L \subset T^*M$ of an F-manifold determines the multiplication on \mathcal{T}_M via the isomorphism ((6.1) and (2.2))

$$\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L, \quad X \mapsto \alpha(\tilde{X})|_L . \quad (15.1)$$

One can generalize this and replace L , T^*M , and α by other spaces and other 1-forms. This allows to find F-manifolds in natural geometric situations and to encode them in an economic way. Corollary 15.2 and Definition 15.4 are the two most interesting special cases of Theorem 15.1.

Theorem 15.1. *Let the following data be given:*
manifolds Z and M , where M is connected and n -dimensional;
a surjective map $\pi_Z : Z \rightarrow M$ which is everywhere a submersion;
an everywhere n -dimensional reduced subvariety $C \subset Z$ such that the
restriction $\pi_C : C \rightarrow M$ is finite;
a 1-form α_Z on Z with the property:

$$\begin{aligned} &\text{any local lift } \tilde{X} \in \mathcal{T}_Z \text{ of the zero vector field } 0 \in \mathcal{T}_M \\ &\text{satisfies } \alpha_Z(\tilde{X})|_C = 0 . \end{aligned} \quad (15.2)$$

Then

a) The map

$$\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C, \quad X \mapsto \alpha_Z(\tilde{X})|_C \quad (15.3)$$

is welldefined; here $\tilde{X} \in \mathcal{T}_Z$ is any lift of X to a neighborhood of C in Z .

b) The image $L \subset T^*M$ of the map

$$\begin{aligned} \mathbf{q} : C &\rightarrow T^*M, \\ z &\mapsto \mathbf{q}(z) = (X \mapsto \mathbf{a}_C(X)(z)) \in T_{\pi_C(z)}^* M \end{aligned} \quad (15.4)$$

is a (reduced) variety. $\mathbf{q} : C \rightarrow L$ is a finite map, the projections $\pi : L \rightarrow M$ and $\pi_C = \pi \circ \mathbf{q}$ are branched coverings. The composition of the maps $\hat{\mathbf{q}} : \pi_* \mathcal{O}_L \rightarrow (\pi_C)_* \mathcal{O}_C$ and

$$\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L, \quad X \mapsto \alpha(\tilde{X})|_L \quad (15.5)$$

is $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$. All three are \mathcal{O}_M -module homomorphisms.

c) The 1-forms α and α_Z satisfy $(\mathbf{q}^* \alpha)|_{C_{reg}} = \alpha_Z|_{C_{reg}}$. Therefore L is a Lagrange variety iff $\alpha_Z|_{C_{reg}}$ is exact.

d) $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ is an isomorphism iff

- i) \mathbf{a}_C is injective,
- ii) its image $\mathbf{a}_C(\mathcal{T}_M) \subset (\pi_C)_* \mathcal{O}_C$ is multiplication invariant,
- iii) $\mathbf{a}_C(\mathcal{T}_M)$ contains the unit $1_C \in (\pi_C)_* \mathcal{O}_C$.

In this case $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ induces a (commutative and associative and) generically semisimple multiplication on \mathcal{T}_M with global unit field and with analytic spectrum L .

e) $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ provides M with the structure of a massive F -manifold iff $\alpha_Z|_{C_{reg}}$ is exact and the conditions i)–iii) in d) are satisfied.

Proof: a) This follows from (15.2).

b) $\dim C = n = \dim M$ and π_C finite imply that π_C is open. M is connected, thus π_C is a branched covering. Using local coordinates for M and T^*M one sees that $\mathbf{q} : C \rightarrow T^*M$ is an analytic map. $\pi_C = \pi \circ \mathbf{q}$

is clear and shows that \mathbf{q} is finite. Then $L = \mathbf{q}(C)$ is a variety and π is a branched covering. $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ follows from the definition of \mathbf{q} .

c) There is an open subset $M^{(0)} \subset M$ with analytic complement $M - M^{(0)}$ such that $\pi_C^{-1}(M^{(0)}) \subset C$ and $\pi^{-1}(M^{(0)}) \subset L$ are smooth, $\pi_C : \pi_C^{-1}(M^{(0)}) \rightarrow M^{(0)}$ and $\pi : \pi^{-1}(M^{(0)}) \rightarrow M^{(0)}$ are coverings and $\mathbf{q} : \pi_C^{-1}(M^{(0)}) \rightarrow \pi^{-1}(M^{(0)})$ is a covering on each component of $\pi^{-1}(M^{(0)})$. Now $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ implies $\mathbf{q}^* \alpha|_{\pi_C^{-1}(M^{(0)})} = \alpha_Z|_{\pi_C^{-1}(M^{(0)})}$.

d) $\hat{\mathbf{q}} : \pi_* \mathcal{O}_L \rightarrow (\pi_C)_* \mathcal{O}_C$ is an injective homomorphism of \mathcal{O}_M -algebras. If $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ is an isomorphism then i)–iii) are obviously satisfied.

Suppose that i)–iii) are satisfied. Then $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ is injective with multiplication invariant image $\mathbf{a}(\mathcal{T}_M) \subset \pi_* \mathcal{O}_L$ and with $1_L \in \mathbf{a}(\mathcal{T}_M)$. The maps \mathbf{a} and \mathbf{a}_C induce the same (commutative and associative) multiplication with global unit field on TM .

We have to show that this multiplication is generically semisimple with analytic spectrum L . Then $\mathbf{a} : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ is an isomorphism and the proof of d) is complete.

If for each $p \in M$ the linear forms in $\pi^{-1}(p) \subset T_p^* M$ would generate a subspace of $T_p^* M$ of dimension $< n$ then \mathbf{a} would not be injective. So, for a generic point $p \in M$ there exist n elements in $\pi^{-1}(p) \subset T_p^* M$ which form a basis of $T_p^* M$. We claim that $\pi^{-1}(p)$ contains no other than these elements: $\pi^{-1}(p)$ does not contain $0 \in T_p^* M$ because of $1_L \in \mathbf{a}(\mathcal{T}_M)$. From the multiplication invariance of $\mathbf{a}(\mathcal{T}_M)$ one derives easily that $\pi^{-1}(p)$ does not contain any further elements.

This extends to a small neighborhood U of the generic point $p \in M$: $\pi^{-1}(U)$ consists of n sheets which form a basis of sections of $T^* M$; the map $\mathbf{a}|_U : \mathcal{T}_U \rightarrow \pi_*(\pi^{-1}(U))$ is an isomorphism and induces a semisimple multiplication on TM with analytic spectrum $\pi^{-1}(U)$.

Then L is the analytic spectrum of the multiplication on \mathcal{T}_M because M is connected.

e) c)+d) and Theorem 6.2. □

In Theorem 15.1 the map $\pi : L \rightarrow M$ has degree n , but $\pi_C : C \rightarrow M$ can have degree $> n$; and even if $\pi_C : C \rightarrow M$ has degree n the map $\mathbf{q} : C \rightarrow L$ does not need to be an isomorphism. Examples will be discussed below (Examples 15.5, Lemma 17.9). But the most important special case is the following.

Corollary 15.2. *Let $Z, M, \pi_Z, C \subset Z, \alpha_Z, \mathbf{a}_C, L$, and \mathbf{q} be as in Theorem 15.1. Suppose that $\alpha_Z|_{C_{reg}}$ is exact and $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ is an isomorphism.*

Then $\mathbf{q} : C \rightarrow L$ is an isomorphism and $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ provides M with the structure of a massive F-manifold with analytic spectrum L .

Proof: Theorem 15.1 e) gives all except for the isomorphism $\mathbf{q} : C \rightarrow L$. This follows from the isomorphism $\hat{\mathbf{q}} : \pi_* \mathcal{O}_L \rightarrow (\pi_C)_* \mathcal{O}_C$ and a universal property of the analytic spectrum. \square

One can encode an irreducible germ of a massive F-manifold with data as in Corollary 15.2 such that the dimension of Z is minimal.

Lemma 15.3. *Let (M, p) be an irreducible germ of a massive n -dimensional F-manifold. $\mathbf{m} \subset T_p M$ denotes the maximal ideal in $T_p M$.*

a) $\dim Z \geq n + \dim \mathbf{m}/\mathbf{m}^2$ for any data as in Corollary 15.2 for (M, p) .

b) There exist data as in Corollary 15.2 for (M, p) with $\dim Z = n + \dim \mathbf{m}/\mathbf{m}^2$ (the construction will be given in the proof).

Proof: a) $\pi_C^{-1}(p) = \pi_Z^{-1}(p) \cap C$ consists of one fat point with structure ring $T_p M$. Its embedding dimension $\dim \mathbf{m}/\mathbf{m}^2$ is bounded by the dimension $\dim \pi_Z^{-1}(p) = \dim Z - n$ of the smooth fiber $\pi_Z^{-1}(p)$.

b) One can choose coordinates $(t_1, \dots, t_n) = (t_1, t') = t$ for (M, p) with $e = \frac{\partial}{\partial t_1}$ as usual and with

$$\bigoplus_{i=2}^n \mathbb{C} \cdot \left[\frac{\partial}{\partial t_i} \right] = \mathbf{m} \subset T_p M \quad \text{and} \quad (15.6)$$

$$\bigoplus_{i=m+1}^n \mathbb{C} \cdot \left[\frac{\partial}{\partial t_i} \right] = \mathbf{m}^2 \subset T_p M \quad (15.7)$$

for $m = 1 + \dim \mathbf{m}/\mathbf{m}^2$. The dual coordinates on $(T^* M, T_p^* M)$ are y_1, \dots, y_n , the analytic spectrum is (cf. (2.1))

$$L = \{(y, t) \mid y_1 = 1, y_i y_j = \sum a_{ij}^k(t') y_k\}. \quad (15.8)$$

Because of (15.7) there exist functions $b_i \in \mathbb{C}\{t'\}[y_2, \dots, y_m]$ with

$$y_i|_L = b_i(y_2, \dots, y_m, t')|_L \quad \text{for } i = m+1, \dots, n. \quad (15.9)$$

We identify (M, p) and $(\mathbb{C}^n, 0)$ using (t_1, \dots, t_n) and define $(Z, 0) = (\mathbb{C}^{m-1} \times \mathbb{C}^n, 0)$. The embedding

$$\iota : (Z, 0) = (\mathbb{C}^{m-1} \times \mathbb{C}^n, 0) \hookrightarrow T^* M, \quad (15.10)$$

$$(x_1, \dots, x_{m-1}, t) \mapsto (y, t) = (1, x_1, \dots, x_{m-1}, b_{m+1}(x, t'), \dots, b_n(x, t'), t)$$

provides canonical choices for the other data,

$$\pi_Z : (Z, 0) \rightarrow (M, p), (x, t) \mapsto t, \quad (15.11)$$

$$C = \iota^{-1}(L), \quad (15.12)$$

$$\alpha_Z = \iota^* \alpha = dt_1 + \sum_{i=2}^m x_{i-1} dt_i + \sum_{i=m+1}^n b_i(x, t') dt_i. \quad (15.13)$$

The conditions in Corollary 15.2 are obviously satisfied. \square

The notion of a generating family for a Lagrange map ([AGV] 19., [Gi2] 1.4) motivates to single out another special case of Theorem 15.1.

Definition 15.4. Let Z, M, π_Z, C, α_Z , and \mathbf{a}_C be as in Theorem 15.1 with $\alpha_Z|_{C_{reg}}$ exact and $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ injective with multiplication invariant image $\mathbf{a}_C(\mathcal{T}_M) \supset \{1_C\}$. These data yield a massive F-manifold (M, \circ, e) .

A function $F : Z \rightarrow \mathbb{C}$ is a *generating family* for this F-manifold if $\alpha_Z = dF$ and if C is the critical set of the map $(F, \pi_Z) : Z \rightarrow \mathbb{C} \times M$.

The name *generating family* has two reasons:

- 1) F is considered as a family of functions on the fibers $\pi_Z^{-1}(p)$, $p \in M$.
- 2) The restriction of F to C is the lift of a generating function $\tilde{F} : L \rightarrow \mathbb{C}$, i.e. $F = \tilde{F} \circ \mathbf{q}$; so the 1-graph of F as a multivalued function on M is L .

In the case of a generating family the conditions (15.2) and α_Z exact are obvious. The most difficult condition is the multiplication invariance of $\mathbf{a}(\mathcal{T}_M)$.

It is not clear whether for any massive F-manifold M data (Z, π_Z, F) as in Definition 15.4 exist. But often even many nonisomorphic data exist. We illustrate this for the 2-dimensional germs $I_2(m)$ of F-manifolds (chapter 11).

Examples 15.5. Always $(Z, 0) = (\mathbb{C} \times \mathbb{C}^2, 0)$ and $(M, p) = (\mathbb{C}^2, 0)$ with projection $\pi_Z : (Z, 0) \rightarrow (M, 0)$, $(x, t_1, t_2) \mapsto (t_1, t_2)$ and $e := \delta_1 := \frac{\partial}{\partial t_1}, \delta_2 := \frac{\partial}{\partial t_2}$.

- a) $C = \{(x, t) \mid x^{m-2} - t_2^2 = 0\}$, $\alpha_Z = dt_1 + x dt_2$.

These are data as in Corollary 15.2 for $I_2(m)$.

- b) Generating family $F = t_1 + \int_0^x (t_2 - u^2)^k du$ ($k \geq 1$),
 $C = \{(x, t) \mid t_2 - x^2 = 0\}$,

$$\alpha_Z|_C = dF|_C = (dt_1 + c \cdot x^{2k-1} dt_2)|_C \text{ for some } c \in \mathbb{C} - \{0\},$$

$$\mathbf{a}_C(\delta_2) \cdot \mathbf{a}_C(\delta_2) = c^2 \cdot t_2^{2k-1} \cdot 1_C.$$

These are data as in Definition 15.4 for $I_2(2k+1)$, $\pi_C : C \rightarrow M$ has degree 2, $\mathbf{q} : C \rightarrow L$ is the normalisation and the maximalisation of L .

c) Generating family $F = t_1 + x^{k+1}t_2 - \frac{k+1}{k+2}x^{k+2}$ ($k \geq 1$),
 $C = \{(x, t) \mid (t_2 - x)x = 0\}$,
 $\alpha_Z|_{C_{reg}} = dF|_{C_{reg}} = (dt_1 + x^{k+1}dt_2)|_{C_{reg}}$,
 $\mathbf{a}_C(\delta_2 - \frac{1}{2}t_2^{k+1} \cdot \delta_1)^2 = \frac{1}{4}t_2^{2k+2} \cdot 1_C$.
 These are data as in Definition 15.4 for $I_2(2k+4)$, $\pi_C : C \rightarrow M$ has degree 2, $\mathbf{q} : C \rightarrow L$ is the maximalisation of L (for the missing case $I_2(4)$ compare Lemma 17.9).

d) Generating family $F = t_1 + \int_0^x (u^2 - t_2)^k u du$ ($k \geq 1$),
 $C = \{(x, t) \mid (t_2 - x^2)x = 0\}$,
 $\alpha_Z|_{C_{reg}} = dF|_{C_{reg}} = (dt_1 + c \cdot x^{2k}dt_2)|_{C_{reg}}$ for some $c \in \mathbb{C} - \{0\}$,
 $\mathbf{a}_C(\delta_2 - \frac{1}{2}ct_2^k \cdot \delta_1)^2 = \frac{1}{4}c^2t_2^{2k} \cdot 1_C$.
 These are data as in Definition 15.4 for $I_2(2k+2)$, $\pi_C : C \rightarrow M$ has degree 3, $\mathbf{q} : C \rightarrow L$ covers one component with degree 1, the other with degree 2.

16. HYPERSURFACE SINGULARITIES

A distinguished class of germs of massive F-manifolds is related to isolated hypersurface singularities: the base space of a semiuniversal unfolding of an isolated hypersurface singularity is an irreducible germ of a massive F-manifold with smooth analytic spectrum (Theorem 16.3). In fact, there is a 1-1 correspondence between such germs of F-manifolds and singularities up to stable right equivalence (Theorem 16.6).

The structure of an F-manifold on the base space has beautiful geometric implications and interpretations (Theorem 16.4, Remarks 16.5). Many of them have been known since long time from different points of view. The concept of an F-manifold unifies them. On the other hand, for much of the general treatment of F-manifolds in this paper the singularity case had been the model case.

An isolated hypersurface singularity is a holomorphic function germ $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity at 0. Its Milnor number $\mu \in \mathbb{N}$ is the dimension of the Jacobi algebra $\mathcal{O}_{\mathbb{C}^m, 0} / (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}) = \mathcal{O}_{\mathbb{C}^m, 0} / J_f$.

The notion of an unfolding of an isolated hypersurface singularity goes back to Thom and Mather. An *unfolding* of f is a holomorphic function germ $F : (\mathbb{C}^m \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ such that $F|_{\mathbb{C}^m \times \{0\}} = f$. We will write the parameter space as $(M, 0) = (\mathbb{C}^n, 0)$.

The *critical space* $(C, 0) \subset (\mathbb{C}^m \times M, 0)$ of the unfolding $F = F(x_1, \dots, x_m, t_1, \dots, t_n)$ is the critical space of the map $(F, pr) : (\mathbb{C}^m \times$

$M, 0) \rightarrow (\mathbb{C} \times M, 0)$. It is the zero set of the ideal

$$J_F := \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m} \right) \quad (16.1)$$

with the complex structure $\mathcal{O}_{C,0} = \mathcal{O}_{\mathbb{C}^m \times M,0} / J_F|_{(C,0)}$.

The intersection $C \cap (\mathbb{C}^m \times \{0\}) = \{0\}$ is a point and $(C, 0)$ is a complete intersection of dimension n . Therefore the projection $pr : (C, 0) \rightarrow (M, 0)$ is finite and flat with degree μ and $\mathcal{O}_{C,0}$ is a free $\mathcal{O}_{M,0}$ -module of rank μ .

A kind of *Kodaira-Spencer map* is the $\mathcal{O}_{M,0}$ -linear map

$$\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}, \quad X \mapsto \tilde{X}(F)|_{(C,0)} \quad (16.2)$$

where \tilde{X} is any lift of $X \in \mathcal{T}_{M,0}$ to $(\mathbb{C}^m \times M, 0)$. Dividing out the submodules $\mathbf{m}_{M,0} \cdot \mathcal{T}_{M,0}$ and $\mathbf{m}_{M,0} \cdot \mathcal{O}_{C,0}$ one obtains the *reduced Kodaira-Spencer map*

$$\mathbf{a}_C|_0 : T_0 M \rightarrow \mathcal{O}_{\mathbb{C}^m,0} / J_f. \quad (16.3)$$

All these objects are independent of the choice of coordinates. In fact, they even behave well with respect to morphisms of unfoldings.

There are several possibilities to define morphisms of unfoldings (cf. Remark 16.2 iv)). We need the following.

Let $F_i : (\mathbb{C}^m \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, be two unfoldings of f with projections $pr_i : (\mathbb{C}^m \times M_i, 0) \rightarrow (M_i, 0)$, critical spaces C_i , and Kodaira-Spencer maps \mathbf{a}_{C_i} . A *morphism* from F_1 to F_2 is a pair (ϕ, ϕ_{base}) of map germs such that the following diagram commutes,

$$\begin{array}{ccc} (\mathbb{C}^m \times M_1, 0) & \xrightarrow{\phi} & (\mathbb{C}^m \times M_2, 0) \\ \downarrow pr_1 & & \downarrow pr_2 \\ (M_1, 0) & \xrightarrow{\phi_{base}} & (M_2, 0), \end{array} \quad (16.4)$$

and

$$\phi|_{\mathbb{C}^m \times \{0\}} = id, \quad (16.5)$$

$$F_1 = F_2 \circ \phi \quad (16.6)$$

hold. One says that F_1 is *induced* by (ϕ, ϕ_{base}) from F_2 .

The definition of critical spaces is compatible with the morphism (ϕ, ϕ_{base}) , that is, $\phi^* J_{F_2} = J_{F_1}$ and $(C_1, 0) = \phi^{-1}((C_2, 0))$. Also the Kodaira-Spencer maps behave well: the $\mathcal{O}_{M_1,0}$ -linear maps

$$d\phi_{base} : \mathcal{T}_{M_1,0} \rightarrow \mathcal{O}_{M_1,0} \otimes_{\mathcal{O}_{M_2,0}} \mathcal{T}_{M_2,0}, \quad (16.7)$$

$$\mathbf{a}_{C_2} : \mathcal{O}_{M_1,0} \otimes_{\mathcal{O}_{M_2,0}} \mathcal{T}_{M_2,0} \rightarrow \mathcal{O}_{M_1,0} \otimes_{\mathcal{O}_{M_2,0}} \mathcal{O}_{C_2,0}, \quad (16.8)$$

$$\phi^*|_{(C_2,0)} : \mathcal{O}_{M_1,0} \otimes_{\mathcal{O}_{M_2,0}} \mathcal{O}_{C_2,0} \rightarrow \mathcal{O}_{C_1,0} \quad (16.9)$$

are defined in the obvious way; their composition is

$$\mathbf{a}_{C_1} = \phi^*|_{(C_2,0)} \circ \mathbf{a}_{C_2} \circ d\phi_{base} . \quad (16.10)$$

(16.9) restricts to the identity on the Jacobi algebra of f because of (16.5). Therefore the reduced Kodaira-Spencer maps satisfy

$$\mathbf{a}_{C_1}|_0 = \mathbf{a}_{C_2}|_0 \circ d\phi_{base}|_0 . \quad (16.11)$$

An unfolding of f is *versal* if any unfolding is induced from it by a suitable morphism. A versal unfolding $F : (\mathbb{C}^m \times M, 0) \rightarrow (\mathbb{C}, 0)$ is *semiuniversal* if the dimension of the parameter space $(M, 0)$ is minimal. Semiuniversal unfoldings of an isolated hypersurface singularity exist by work of Thom and Mather. Detailed proofs can nowadays be found at many places, e.g. [Was][AGV](8.).

Theorem 16.1. *An unfolding $F : (\mathbb{C}^m \times M, 0) \rightarrow (\mathbb{C}, 0)$ of an isolated hypersurface singularity $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ is versal iff the reduced Kodaira-Spencer map $\mathbf{a}_C|_0 : T_0M \rightarrow \mathcal{O}_{\mathbb{C}^m,0}/J_f$ is surjective. It is semiuniversal iff $\mathbf{a}_C|_0$ is an isomorphism.*

Remarks 16.2. i) Because of the lemma of Nakayama $\mathbf{a}_C|_0$ is surjective (an isomorphism) iff \mathbf{a}_C is surjective (an isomorphism).

ii) A convenient choice of a semiuniversal unfolding $F : (\mathbb{C}^m \times \mathbb{C}^\mu, 0) \rightarrow (\mathbb{C}, 0)$ is $F(x_1, \dots, x_m, t_1, \dots, t_\mu) = f + \sum_{i=1}^\mu m_i t_i$, where $m_1, \dots, m_\mu \in \mathcal{O}_{\mathbb{C}^m,0}$ represent a basis of the Jacobi algebra of f , preferably with $m_1 = 1$.

iii) The critical space of an unfolding $F : (\mathbb{C}^m \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is reduced and smooth iff the matrix $(\frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial^2 F}{\partial x_i \partial t_k})(0)$ has maximal rank m . This is satisfied for versal unfoldings.

iv) In the literature (e.g. [Was]) one often finds a slightly different notion of morphisms of unfoldings: An (r) -*morphism* between unfoldings F_1 and F_2 as above is a triple $(\phi, \phi_{base}, \tau)$ of map germs ϕ and ϕ_{base} with (16.4) and (16.5) and $\tau : (M_1, 0) \rightarrow (\mathbb{C}, 0)$ with (16.6) replaced by

$$F_1 = F_2 \circ \phi + \tau . \quad (16.12)$$

(r) -*versal* and (r) -*semiuniversal* unfoldings are defined analogously. They exist ([Was]): an unfolding $F : (\mathbb{C}^m \times M, 0) \rightarrow (\mathbb{C}, 0)$ is (r) -versal ((r) -semiuniversal) iff the map

$$\mathbb{C} \oplus T_0M \rightarrow \mathcal{O}_{\mathbb{C}^m,0}/J_f, (c, X) \mapsto c + \mathbf{a}_C|_0(X) \quad (16.13)$$

is surjective (an isomorphism).

So one gains a bit: the base space of an (r) -semiuniversal unfolding $F^{(r)}$ has dimension $\mu - 1$; if $F^{(r)} = F^{(r)}(x_1, \dots, x_m, t_2, \dots, t_\mu)$ is (r) -semiuniversal then $t_1 + F^{(r)}$ is semiuniversal; between two semiuniversal

unfoldings $t_1 + F_1^{(r)}$ and $t_1 + F_2^{(r)}$ of this form there exist isomorphisms which come from (r) -isomorphisms of $F_1^{(r)}$ and $F_2^{(r)}$. (The relation between $F^{(r)}$ and $t_1 + F^{(r)}$ motivates the “ (r) ”, which stands for “restricted”).

On the other hand, one loses (16.10). Anyway, one should keep (r) -semiuniversal unfoldings in mind. They are closely related to miniversal Lagrange maps (see the proof of Theorem 16.6 and [AGV] 19.).

v) One can generalize the notion of a morphism between unfoldings if one weakens the condition (16.5): Let $F_i : (\mathbb{C}^m \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, be unfoldings of two isolated hypersurface singularities f_1 and f_2 . A *generalized morphism* from F_1 to F_2 is a pair (ϕ, ϕ_{base}) of map germs with a commutative diagram as in (16.4) such that (16.6) holds and $\phi|_{\mathbb{C}^m \times \{0\}}$ is a coordinate change (between f_1 and f_2).

Then f_1 and f_2 are *right equivalent*. If the generalized morphism is invertible then also F_1 and F_2 are called *right equivalent*.

Critical spaces and Kodaira-Spencer maps behave well also for generalized morphisms; (16.10) holds, in (16.11) one has to take into account the isomorphism of the Jacobi algebras of f_1 and f_2 which is induced by $\phi|_{\mathbb{C}^m \times \{0\}}$.

The multiplication on the base space of a semiuniversal unfolding was first defined by K. Saito [SaK4](1.5)[SaK5](1.3).

Theorem 16.3. *Let $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated hypersurface singularity and $F : (\mathbb{C}^m \times M, 0) \rightarrow (\mathbb{C}, 0)$ a semiuniversal unfolding.*

The Kodaira-Spencer map $\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{\mathbb{C},0}$ is an isomorphism and induces a multiplication on $\mathcal{T}_{M,0}$. Then $(M, 0)$ is an irreducible germ of a massive F -manifold with smooth analytic spectrum. $E := \mathbf{a}_C^{-1}(F|_C)$ is an Euler field of weight 1.

Proof: $\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{\mathbb{C},0}$ is an isomorphism because of Theorem 16.1 and Remark 16.2 i). The critical space $(C, 0)$ is reduced and smooth. One applies Corollary 15.2 to $(Z, 0) = (\mathbb{C}^m \times M, 0)$ and $\alpha_Z = dF$. The map $\mathbf{q} : (C, 0) \rightarrow (L, \pi^{-1}(0))$ is an isomorphism. $\pi^{-1}(0)$ is a point. Theorem 15.1 c) shows that $F|_C \circ \mathbf{q}^{-1}$ is a holomorphic generating function. Therefore E is an Euler field of weight 1. \square

Theorem 16.4. *Let $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated hypersurface singularity and $F_i : (\mathbb{C}^m \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, be two semiuniversal unfoldings.*

There exists a unique isomorphism $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ of F -manifolds such that $\phi_{base} = \varphi$ for any isomorphism (ϕ, ϕ_{base}) of the unfoldings F_1 and F_2 .

Proof: $\phi_{base} : (M_1, 0) \rightarrow (M_2, 0)$ is an isomorphism of F-manifolds because of (16.10).

Suppose that $F_1 = F_2$ and $(M_1, 0) = (M_2, 0)$. The tangent map of ϕ_{base} on $T_0 M_1$ is $d\phi_{base}|_0 = id$ because of (16.11). The group of all automorphisms of $(M_1, 0)$ as F-manifold is finite (Theorem 14.1). Therefore $\phi_{base} = id$. \square

Remarks 16.5. i) The rigidity of the base morphism ϕ_{base} in Theorem 16.4 is in sharp contrast to the general situation for deformations of geometric objects. Usually only a part of the base space of a miniversal deformation is rigid with respect to automorphisms of the deformation.

ii) The reason for the rigidity are, via Theorem 14.1 and Theorem 10.2, the canonical coordinates at generic parameters. The corresponding result for singularities is that the critical values of F form coordinates on the base at generic parameters. It had been proved by Looijenga [Lo1].

iii) Because of this rigidity also the openness of versality (e.g. [Tei]) takes a special form: For any point $t \in M$ in a representative of the base space $(M, 0) = (\mathbb{C}^\mu, 0)$ of a semiuniversal unfolding F , Theorem 4.2 yields a unique decomposition $(M, t) = \prod_{k=1}^l (M_k, t)$ into a product of irreducible germs of F-manifolds. These germs (M_k, t) are the base spaces of semiuniversal unfoldings of the singularities of $F|_{\mathbb{C}^m \times \{t\}}$. The multigerm of F at $\mathbb{C}^m \times \{t\} \cap C$ itself is isomorphic – in a way which can be made precise easily – to a transversal union of versal unfoldings of these singularities.

iv) The tangent space $T_t M \cong \bigoplus_{k=1}^l T_t M_k$ is canonically isomorphic to the direct sum of the Jacobi algebras of singularities of $F|_{\mathbb{C}^m \times \{t\}}$. The vector in $T_t M$ of the Euler field E is mapped to the direct sum of the classes of the function $F|_{\mathbb{C}^m \times \{t\}}$ in these Jacobi algebras. A result of Scherk [Sch] says:

The Jacobi algebra $\mathcal{O}_{\mathbb{C}^m, 0}/J_f$ of an isolated hypersurface singularity $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ together with the class $[f] \in \mathcal{O}_{\mathbb{C}^m, 0}/J_f$ determines f up to right equivalence.

This result shows that the base space M as an F-manifold with Euler field E determines for each parameter $t \in M$ the singularities of $F|_{\mathbb{C}^m \times \{t\}}$ up to right equivalence and also the critical values. Theorem 16.6 will give an even stronger result.

v) The eigenvalues of $E \circ : T_t M \rightarrow T_t M$ are the critical values of $F|_{\mathbb{C}^m \times \{t\}}$. Therefore the discriminant of the Euler field E is

$$\mathcal{D} = \{t \in M \mid (\det(E \circ))(t) = 0\} = \pi_C(C \cap F^{-1}(0)) \quad (16.14)$$

and coincides with the classical discriminant of the unfolding F .

All the results of chapter 11 apply to this discriminant. Of course, many of them are classic in the singularity case.

For example, Theorem 11.1 and the isomorphism $\mathbf{q} : C \rightarrow L$ from Corollary 15.2 yield an isomorphism between the development $\tilde{\mathcal{D}} \subset \mathbb{P}T^*M$ of the discriminant and the smooth variety $C \cap F^{-1}(0)$ which had been established by Teissier [Tei]. Implicitly it is also in [AGV](19.).

The elementary way in Corollary 11.6 how discriminant and unit field determine the Jacobi algebras seems to be new. But the consequence from this and Scherk's result that discriminant and unit field determine the singularity (up to right equivalence) is known (compare below Theorem 16.6 and Remark 16.7 iv)).

Arnold studied the relation between singularities and Lagrange maps [AGV](19.). His results (cf. also [Ph1](4.7.4.1, pp 299–301), [Ph2], [Wi](Corollary 10)) together with those of chapter 9 yield the following correspondence between unfoldings and certain germs of F-manifolds.

Theorem 16.6. *a) Each irreducible germ of a massive F-manifold with smooth analytic spectrum is the base space of a semiuniversal unfolding of an isolated hypersurface singularity.*

b) Suppose, $F_i : (\mathbb{C}^{m_i} \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, are semiuniversal unfoldings of singularities $f_i : (\mathbb{C}^{m_i}, 0) \rightarrow (\mathbb{C}, 0)$ and $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ is an isomorphism of the base spaces as F-manifolds. Suppose that $m_1 \leq m_2$.

Then a coordinate change $\psi : (\mathbb{C}^{m_2}, 0) \rightarrow (\mathbb{C}^{m_2}, 0)$ exists such that

$$f_1(x_1, \dots, x_{m_1}) + x_{m_1+1}^2 + \dots + x_{m_2}^2 = f_2(x_1, \dots, x_{m_2}) \circ \psi \quad (16.15)$$

and an isomorphism (ϕ, ϕ_{base}) of the unfoldings $F_1 + x_{m_1+1}^2 + \dots + x_{m_2}^2$ and $F_2 \circ \psi$ exists with

$$F_1 + x_{m_1+1}^2 + \dots + x_{m_2}^2 = F_2 \circ \psi \circ \phi \quad \text{and} \quad \phi_{base} = \varphi. \quad (16.16)$$

Proof: a) The restricted Lagrange map of the germ of a massive F-manifold with smooth analytic spectrum is a miniversal germ of a Lagrange map with smooth Lagrange variety (chapter 9). Arnold [AGV](19.3) constructs a generating family $F^{(r)} = F^{(r)}(x, t_2, \dots, t_\mu)$ for it. Looking at the notions of stable maps and generating families in [AGV](19.), one sees: $F^{(r)}$ is an (r) -semiuniversal unfolding of $F^{(r)}(x, 0)$ (cf. Remark 16.2 iv)). $t_1 + F^{(r)}$ is a semiuniversal unfolding of $F^{(r)}(x, 0)$. Its base space is the given germ of a massive F-manifold.

b) The unfolding F_i is isomorphic to an unfolding $t_1 + F_i^{(r)}(x_1, \dots, x_{m_i}, t_2, \dots, t_\mu)$ as in Remark 16.2 iv) over the same base.

Then $F_i^{(r)}$ is an (r) -semiuniversal unfolding and a generating family for the restricted Lagrange map of the F-manifold $(M_i, 0)$.

The isomorphism $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ induces an isomorphism of the restricted Lagrange maps. Then the main result in [AGV](19.4) establishes a notion of equivalence for $F_1^{(r)}$ and $F_2^{(r)}$, stable \mathcal{R}^+ -equivalence, which yields the desired equivalence in Theorem 16.6 b) for F_1 and F_2 . \square

Remarks 16.7. i) Two isolated hypersurface singularities $f_i : (\mathbb{C}^{m_i}, 0) \rightarrow (\mathbb{C}, 0)$ with $m_1 \leq m_2$ are *stably right equivalent* if a coordinate change $\psi : (\mathbb{C}^{m_2}, 0) \rightarrow (\mathbb{C}^{m_2}, 0)$ with (16.15) exists. They are right equivalent if furthermore $m_1 = m_2$. The splitting lemma says:

An isolated hypersurface singularity $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ with $r := m - \text{rank}(\frac{\partial^2 f}{\partial x_i \partial x_j})(0)$ is stably right equivalent to a singularity $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}, 0)$ with $\text{rank}(\frac{\partial^2 g}{\partial x_i \partial x_j})(0) = 0$; this singularity g is unique up to right equivalence.

(For the existence of g see e.g. [Sl](4.2 Satz), the uniqueness of g up to right equivalence follows from Theorem 16.6 or from Scherk's result (Remark 16.5 iv)).)

ii) Theorem 16.6 gives a 1-1 correspondence between isolated hypersurface singularities up to stable right equivalence and irreducible germs of massive F-manifolds with smooth analytic spectrum.

But the liftability of an isomorphism $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ to unfoldings which is formulated in Theorem 16.6 b) is stronger. The 1-1 correspondence itself follows already from Theorem 16.6 a) and Scherk's result (Remark 16.5 iv)).

iii) The proof of Theorem 16.6 a) is not so difficult. If $(M, 0)$ is an irreducible germ of a massive F-manifold with analytic spectrum $(L, \lambda) \subset T^*M$, then a sufficiently generic extension of a generating function on (L, λ) to a function on (T^*M, λ) is already a semiuniversal unfolding over $(M, 0)$. A version different from [AGV](19.3) of the precise construction is given by Pham [Ph1](4.7.4.1, pp 291–301), following Hörmander.

iv) Theorem 16.6 b) follows also from [Ph2] (again following Hörmander) and from [Wi](Corollary 10). To apply Wirthmüllers arguments one has to start with the discriminant \mathcal{D} and the unit field. Pham [Ph1][Ph2] starts with the *characteristic variety*. That is the cone in T^*M of the development $\tilde{\mathcal{D}} \subset \mathbb{P}T^*M$ of the discriminant.

A semiuniversal unfolding $F : (\mathbb{C}^m \times M, 0) \rightarrow (\mathbb{C}, 0)$ yields data as in Corollary 15.2 for the germ $(M, 0)$ of an F-manifold:

$$(Z, 0) = (\mathbb{C}^m \times M, 0), \quad \alpha_Z = dF. \quad (16.17)$$

The semiuniversal unfolding F is also a generating family of $(M, 0)$ as germ of an F-manifold in the sense of Definition 15.4.

The following observation says that these two special cases Corollary 15.2 and Definition 15.4 of the general construction of F-manifolds in Theorem 15.1 meet only in the case of unfoldings of isolated hypersurface singularities.

Lemma 16.8. *Let $Z, M, \pi_Z, C, \alpha_Z, \mathbf{a}_C$, and $F : Z \rightarrow C$ satisfy all the properties in Corollary 15.2 and Definition 15.4.*

Then C is smooth. For any point $p \in M$ the multigerms $F : (Z, C \cap \pi_Z^{-1}(p)) \rightarrow \mathbb{C}$ is isomorphic to a transversal product of versal unfoldings of the singularities of $F|_{\pi_Z^{-1}(p)}$ (cf. Remark 16.5 iii)). The irreducible germs (M_k, p) of F-manifolds in the decomposition $(M, p) = \prod_{k=1}^l (M_k, p)$ are base spaces of semiuniversal unfoldings of the singularities of $F|_{\pi_Z^{-1}(p)}$.

Proof: The isomorphism $\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C$ of Corollary 15.2 restricts at $p \in M$ to a componentwise isomorphism of algebras

$$T_p M = \bigoplus_{k=1}^l T_p M_k \rightarrow \bigoplus_{z \in C \cap \pi_Z^{-1}(p)} (\text{Jacobi algebra of } F|_{(\pi_Z^{-1}(p), z)}).$$

One applies Theorem 16.1. □

17. BOUNDARY SINGULARITIES

The last chapter showed that germs of F-manifolds with smooth analytic spectrum correspond to isolated hypersurface singularities. The simplest nonsmooth germ of an analytic spectrum of dimension n is isomorphic to

$$(\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^{n-1}, 0).$$

We will see that irreducible germs of massive F-manifolds with such an analytic spectrum correspond to boundary singularities (Theorem 17.6). Boundary singularities had been introduced by Arnold [Ar2]. Because of the similarities to hypersurface singularities we will develop things parallel to chapter 16.

We consider a germ $(\mathbb{C}^{m+1}, 0)$ with coordinates x_0, \dots, x_m always together with the hyperplane $H := \{x \in \mathbb{C}^{m+1} \mid x_0 = 0\}$ of the first coordinate. A *boundary singularity* (f, H) is a holomorphic function

germ $f : (\mathbb{C}^{m+1}, 0) \rightarrow (\mathbb{C}, 0)$ such that f and $f|_H$ have isolated singularities at 0. It can be considered as an extension of the hypersurface singularities f and $f|_H$.

Its Jacobi algebra is

$$\mathcal{O}_{\mathbb{C}^{m+1},0}/J_{f,H} := \mathcal{O}_{\mathbb{C}^{m+1},0}/\left(x_0 \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}\right), \quad (17.1)$$

its Milnor number $\mu = \mu(f, H) := \dim \mathcal{O}_{\mathbb{C}^{m+1},0}/J_{f,H}$ satisfies ([Ar2](§3), [Sz](§2))

$$\mu = \mu(f) + \mu(f|_H). \quad (17.2)$$

An unfolding of (f, H) is simply a holomorphic function germ $F : (\mathbb{C}^{m+1} \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ such that $F|_{\mathbb{C}^{m+1} \times \{0\}} = f$, that is, an unfolding of f . Again we write the parameter space as $(M, 0) = (\mathbb{C}^n, 0)$.

But the critical space $(C, 0) \subset (\mathbb{C}^{m+1} \times M, 0)$ of F as unfolding of the boundary singularity (f, H) is the zero set of the ideal

$$J_{F,H} := \left(x_0 \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m}\right) \quad (17.3)$$

with the complex structure $\mathcal{O}_{C,0} = \mathcal{O}_{\mathbb{C}^{m+1} \times M,0}/J_{F,H}|_{(C,0)}$ (cf. [Sz]). Forgetting the complex structure, $(C, 0)$ is the union of the critical sets $(C^{(1)}, 0)$ of F and $(C^{(2)}, 0)$ of $F|_{H \times M}$ as unfoldings of hypersurface singularities.

For the same reasons as in the hypersurface case the projection $pr : (C, 0) \rightarrow (M, 0)$ is finite and flat with degree μ and $\mathcal{O}_{C,0}$ is a free $\mathcal{O}_{M,0}$ -module of rank μ . The 1-form

$$\alpha_Z := -\frac{\partial F}{\partial x_0} dx_0 + dF = \sum_{i=1}^m \frac{\partial F}{\partial x_i} dx_i + \sum_{j=1}^n \frac{\partial F}{\partial t_j} dt_j \quad (17.4)$$

on $(Z, 0) := (\mathbb{C}^{m+1} \times M, 0)$ gives rise to a kind of Kodaira-Spencer map

$$\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}, \quad X \mapsto \alpha_Z(\tilde{X})|_{(C,0)}, \quad (17.5)$$

where \tilde{X} is any lift of $X \in \mathcal{T}_{M,0}$ to $(Z, 0)$. It induces a reduced Kodaira-Spencer map

$$\mathbf{a}_C|_0 : T_0 M \rightarrow \mathcal{O}_{\mathbb{C}^{m+1},0}/J_{f,H}. \quad (17.6)$$

The ideal $J_{F,H}$ and the maps \mathbf{a}_C and $\mathbf{a}_C|_0$ behave well with respect to morphisms of unfoldings, as we will see.

A morphism between two unfoldings F_1 and F_2 as in chapter 16 of a boundary singularity (f, H) is a pair (ϕ, ϕ_{base}) of a map germ with (16.4)–(16.6) and additionally

$$\phi(H \times M_1) \subset H \times M_2. \quad (17.7)$$

Then the first entry of ϕ takes the form $x_0 \cdot \text{unit} \in \mathcal{O}_{Z,0}$. Using this one can see with a bit more work than in the hypersurface case that the critical spaces behave well with respect to morphisms:

$$\phi^* J_{F_2} = J_{F_1} \quad \text{and} \quad (C_1, 0) = \phi^{-1}((C_2, 0)) . \quad (17.8)$$

Also the Kodaira-Spencer maps behave as well as in the hypersurface case. The $\mathcal{O}_{M_1,0}$ -linear maps $d\phi_{base}$, \mathbf{a}_{C_2} , and $\phi^*|_{(C_2,0)}$ are defined as in (16.7)–(16.9); one finds again

$$\mathbf{a}_{C_1} = \phi^*|_{(C_2,0)} \circ \mathbf{a}_{C_2} \circ d\phi_{base} \quad (17.9)$$

and

$$\mathbf{a}_{C_1}|_0 = \mathbf{a}_{C_2}|_0 \circ d\phi_{base}|_0 . \quad (17.10)$$

Versal and semiuniversal unfoldings of boundary singularities are defined analogously to the hypersurface case and exist.

Theorem 17.1 ([Ar2]). *An unfolding $F : (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ of a boundary singularity (f, H) , $f : (\mathbb{C}^{m+1}, 0) \rightarrow (\mathbb{C}, 0)$, is versal iff the reduced Kodaira-Spencer map $\mathbf{a}_C|_0 : T_0 M \rightarrow \mathcal{O}_{\mathbb{C}^{m+1},0}/J_{f,H}$ is surjective. It is semiuniversal iff $\mathbf{a}_C|_0$ is an isomorphism.*

Remarks 17.2. i) $\mathbf{a}_C|_0$ is surjective (an isomorphism) iff \mathbf{a}_C is surjective (an isomorphism).

ii) $F(x_0, \dots, x_m, t_1, \dots, t_\mu) = f + \sum_{i=1}^\mu m_i t_i$ is a semiuniversal unfolding of the boundary singularity (f, H) if $m_1, \dots, m_\mu \in \mathcal{O}_{\mathbb{C}^{m+1},0}$ represent a basis of $\mathcal{O}_{\mathbb{C}^{m+1},0}/J_{f,H}$.

iii) The critical space of an unfolding $F : (Z, 0) = (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ of a boundary singularity (f, H) is reduced and isomorphic to $(\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^{n-1}, 0)$ iff $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_m}$ represent a generating system of the vector space $\mathbf{m}_{Z,0}/((x_0) + \mathbf{m}_{Z,0}^2)$. This is equivalent to the nondegeneracy condition

$$\text{rank} \left(\begin{array}{cc} \frac{\partial^2 F}{\partial x_0 \partial x_j} & \frac{\partial^2 F}{\partial x_0 \partial t_k} \\ \frac{\partial^2 F}{\partial x_i \partial x_j} & \frac{\partial^2 F}{\partial x_i \partial t_k} \end{array} \right)_{i,j \geq 1} (0) = m + 1 \quad (17.11)$$

(cf. [NN]). It is satisfied for versal unfoldings.

iv) As in Remark 16.2 v) for hypersurface singularities, one can define generalized morphisms between unfoldings of right equivalent boundary singularities. Again the critical spaces and Kodaira-Spencer maps behave well.

Theorem 17.3. *Let $F : (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ be a semiuniversal unfolding of a boundary singularity (f, H) .*

The Kodaira-Spencer map $\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}$ is an isomorphism and induces a multiplication on $\mathcal{T}_{M,0}$. Then $(M, 0)$ is an irreducible germ of a massive F -manifold with analytic spectrum isomorphic to $(\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^{\mu-1}, 0)$. The field $E := \mathbf{a}_C(F|_C)$ is an Euler field of weight 1.

Proof: Similar to the proof of Theorem 16.3. One wants to apply Corollary 15.2 and has to show that $\alpha_Z|_{C_{reg}}$ is exact. The critical space $(C, 0)$ as a set is the union of the smooth zero sets $(C^{(1)}, 0)$ of J_F and $(C^{(2)}, 0)$ of the ideal $(x_0, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m})$. The definition (17.4) of α_Z shows

$$\alpha_Z|_{(C^{(i)}, 0)} = dF|_{(C^{(i)}, 0)} \quad \text{for } i = 1, 2. \quad (17.12)$$

Therefore $\alpha_Z|_{C_{reg}}$ is exact and $F|_C \circ \mathbf{q}^{-1}$ is a holomorphic generating function of the analytic spectrum. \square

Theorem 17.4. *Let $F_i : (\mathbb{C}^{m+1} \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, be two semiuniversal unfoldings of a boundary singularity (f, H) .*

There exists a unique isomorphism $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ of F -manifolds such that $\phi_{base} = \varphi$ for any isomorphism (ϕ, ϕ_{base}) of the unfoldings F_1 and F_2 .

Proof: Similar to the proof of Theorem 16.4. \square

Remarks 17.5. i) Let $F : (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ be a semiuniversal unfolding of a boundary singularity (f, H) with critical space $(C, 0) = (C^{(1)}, 0) \cup (C^{(2)}, 0)$. For any $t \in M$ the points in $\mathbb{C}^{m+1} \times \{t\} \cap (C, 0)$ split into three classes: The hypersurface singularities of $F|_{\mathbb{C}^{m+1} \times \{t\}}$ in $C^{(1)} - C^{(2)}$, the hypersurface singularities of $F|_{H \times \{t\}}$ in $C^{(2)} - C^{(1)}$, and the boundary singularities of $F|_{\mathbb{C}^{m+1} \times \{t\}}$ in $C^{(1)} \cap C^{(2)}$.

The algebra $\mathcal{O}_C|_{\mathbb{C}^{m+1} \times \{0\}}$ is the direct sum of their Jacobi algebras. The reduced Kodaira-Spencer map at $t \in M$ is an isomorphism from $T_t M$ to this algebra.

Hence the multigerms of F at $\mathbb{C}^{m+1} \times \{t\} \cap C^{(1)}$ and of $F|_{H \times M}$ at $H \times \{t\} \cap (C^{(2)} - C^{(1)})$ together form a transversal union of versal unfoldings of these hypersurface and boundary singularities.

The components (M_k, t) of the decomposition $(M, t) = \prod_{k=1}^l (M_k, t)$ into irreducible germs of F -manifolds are bases of semiuniversal unfoldings of the hypersurface and boundary singularities.

ii) The eigenvalues of $E \circ : T_t M \rightarrow T_t M$ are by definition of E the values of F on $\mathbb{C}^{m+1} \times \{t\} \cap C$. The discriminant of the Euler field is

$$\mathcal{D} = \{t \in M \mid (\det(E \circ))(t) = 0\} = \pi_C(C \cap F^{-1}(0)). \quad (17.13)$$

It is the union of the discriminants of F and $F|H \times M$ as unfoldings of hypersurface singularities and coincides with the classical discriminant of F as unfolding of a boundary singularity [Ar2][Sz]. All the results of chapter 11 apply to this discriminant.

Nguyen huu Duc and Nguyen tien Dai studied the relation between boundary singularities and Lagrange maps [NN]. Their results together with chapter 9 yield the following correspondence between unfoldings of boundary singularities and certain germs of F-manifolds.

Theorem 17.6. *Let $(M, 0)$ be an irreducible germ of a massive F-manifold with analytic spectrum (L, λ) isomorphic to*

$$(\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^{n-1}, 0)$$

and ordered components $(L^{(1)}, \lambda) \cup (L^{(2)}, \lambda) = (L, \lambda)$.

a) There exists a semiuniversal unfolding F of a boundary singularity such that the base space is isomorphic to $(M, 0)$ as F-manifold and the isomorphism $\mathbf{q} : (C, 0) \rightarrow (L, \lambda)$ maps $C^{(i)}$ to $L^{(i)}$.

b) Suppose, $F_i : (\mathbb{C}^{m_i+1} \times M_i, 0) \rightarrow (\mathbb{C}, 0)$, $i = 1, 2$, are semiuniversal unfoldings of boundary singularities (f_i, H_i) and $\varphi : (M_1, 0) \rightarrow (M_2, 0)$ is an isomorphism of the base spaces as F-manifolds. Suppose that $m_1 \leq m_2$.

Then a coordinate change $\psi : (\mathbb{C}^{m_2+1}, 0) \rightarrow (\mathbb{C}^{m_2+1}, 0)$ with $\psi((H_2, 0)) = (H_2, 0)$ exists such that

$$f_1(x_0, \dots, x_{m_1}) + x_{m_1+1}^2 + \dots + x_{m_2}^2 = f_2(x_0, \dots, x_{m_2}) \circ \psi \quad (17.14)$$

and an isomorphism (ϕ, ϕ_{base}) of the unfoldings $F_1 + x_{m_1+1}^2 + \dots + x_{m_2}^2$ and $F_2 \circ \psi$ of boundary singularities exists with

$$F_1 + x_{m_1+1}^2 + \dots + x_{m_2}^2 = F_2 \circ \psi \circ \phi \quad \text{and} \quad \phi_{base} = \varphi. \quad (17.15)$$

Proof: a) In [NN] (Proposition 1) an unfolding $F : (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ with nondegeneracy condition (17.11) of a boundary singularity is constructed such that F is a generating family for $L^{(1)} \subset T^*M$ and $F|H \times M$ is a generating family for $L^{(2)}$.

One can show that there are canonical maps $C^{(i)} \rightarrow L^{(i)}$ which combine to an isomorphism $\mathbf{q} : C \rightarrow L$ with $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ (as in Theorem 15.1). Then the Kodaira-Spencer map $\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}$ is an isomorphism and F is a semiuniversal unfolding of a boundary singularity. (Implicitly this is also contained in [NN] (Théorème)). Because of $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ its base is $(M, 0)$ as F-manifold.

b) [NN] (Proposition 3). □

Remarks 17.7. i) Two boundary singularities $f_i : (\mathbb{C}^{m_i+1}, 0) \rightarrow (\mathbb{C}, 0)$ with $m_1 \leq m_2$ are *stably right equivalent* if a coordinate change ψ as

in Theorem 17.6 b) exists. They are right equivalent if furthermore $m_1 = m_2$. A splitting lemma for boundary singularities is formulated below in Lemma 17.8.

ii) Theorem 17.6 gives a 1-1 correspondence between boundary singularities up to stable right equivalence and irreducible germs of massive F-manifolds with analytic spectrum $(L, \lambda) \cong (\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^2, 0)$ and ordered components $(L^{(1)}, \lambda) \cup (L^{(2)}, \lambda) = (L, \lambda)$.

iii) Interchanging the two components of (L, λ) corresponds to a duality for boundary singularities which goes much further and has been studied by I. Shcherbak, A. Szpirglas [Sz][SS1][SS2], and others.

Lemma 17.8. *(Splitting lemma for boundary singularities)*

A boundary singularity (f, H) with $f : (\mathbb{C}^{m+1}, 0) \rightarrow (\mathbb{C}, 0)$ and $H = \{x \mid x_0 = 0\}$ is stably right equivalent to a boundary singularity $g : (\mathbb{C}^{r+1}, 0) \rightarrow (\mathbb{C}, 0)$ in

$$r + 1 = \max(2; m + 1 - \text{rank}\left(\frac{\partial^2 f(x_0^2, x_1, \dots, x_m)}{\partial x_i \partial x_j}\right)(0)) \quad (17.16)$$

coordinates. The boundary singularity g is unique up to right equivalence.

Proof: Existence of g : The group $G = \mathbb{Z}_2$ acts on $(\mathbb{C}^{m+1}, 0)$ by $(x_0, x_1, \dots, x_m) \mapsto (\pm x_0, x_1, \dots, x_m)$. Boundary singularities on the quotient $(\mathbb{C}^{m+1}, 0)$ correspond to G -invariant singularities on the double cover, branched along H ([AGV] 17.4).

One applies an equivariant splitting lemma of Slodowy [Sl](4.2 Satz) to the G -invariant singularity $f(x_0^2, x_1, \dots, x_m)$. The nondegenerate quadratic part of the G -invariant singularity in splitted form does not contain x_0^2 because f is not smooth.

Uniqueness of g : This follows with Theorem 17.6 b). \square

The following two observations give some information on generating families in the sense of Definition 15.4 for the F-manifolds of boundary singularities. The first one gives a distinguished generating family and is essentially well known. The second one explains why $B_2 = I_2(4)$ is missing in Example 15.5 b).

Lemma 17.9. *a) Let $F : (Z, 0) = (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ be a semi-universal unfolding of a boundary singularity (f, H) .*

Then the function $\tilde{F} : (\tilde{Z}, 0) = (\mathbb{C}^{m+1} \times M, 0) \rightarrow (\mathbb{C}, 0)$ with $\tilde{F}(x, t) = F(x_0^2, x_1, \dots, x_m, t)$ is a generating family for the germ $(M, 0)$ of an F-manifold.

The finite map $\tilde{\mathbf{q}} : \tilde{C} \rightarrow L = L^{(1)} \cup L^{(2)}$ from its critical set \tilde{C} to the analytic spectrum L has degree 2 on $L^{(1)}$ and degree 1 on $L^{(2)}$. The branched covering $\tilde{C} \rightarrow M$ has degree $2\mu(f) + \mu(f|_H)$.

b) Let $(M, 0)$ be a germ of a massive F -manifold with analytic spectrum $(L, \lambda) \cong (\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0) \times (\mathbb{C}^{n-1}, 0)$. There does not exist a generating family $F : (Z, 0) \rightarrow (\mathbb{C}, 0)$ with critical set C such that the canonical map $\mathbf{q} : C \rightarrow L$ is a homeomorphism.

Proof: a) Consider the branched covering $\pi_G : \tilde{Z} \rightarrow Z$, $(x_0, \dots, x_m, t) \mapsto (x_0^2, x_1, \dots, x_m, t)$ which is induced by the action $(x_0, \dots, x_m, t) \mapsto (\pm x_0, x_1, \dots, x_m, t)$ of the group $G = \mathbb{Z}_2$ on \tilde{Z} . The composition $\tilde{F} = F \circ \pi_G$ is an unfolding of the G -invariant singularity $\tilde{F}|(\mathbb{C}^{m+1} \times \{0\}, 0)$, in fact, semiuniversal within the G -invariant unfoldings (cf. [Sl](4.5)). The ideals

$$J_{\tilde{F}} = (2x_0 \cdot (\frac{\partial F}{\partial x_0} \circ \pi_G), \frac{\partial F}{\partial x_1} \circ \pi_G, \dots, \frac{\partial F}{\partial x_m} \circ \pi_G) \quad (17.17)$$

and $\pi_G^* J_{F,H}$ have the same zero sets $\tilde{C} = \pi_G^{-1}(C)$.

Comparison of $\alpha_{\tilde{Z}} = d\tilde{F}$ on \tilde{Z} and $\alpha_Z = -\frac{\partial F}{\partial x_0} dx_0 + dF$ on Z (formula (17.4)) shows that the map $\mathbf{a}_{\tilde{C}} : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{\tilde{C},0}$ factorizes into the Kodaira-Spencer map $\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}$ and the map $\pi_G^*|_{(C,0)} : \mathcal{O}_{C,0} \rightarrow \mathcal{O}_{\tilde{C},0}$.

Therefore $\mathbf{a}_{\tilde{C}}$ is injective with multiplication invariant image and induces the correct multiplication on $\mathcal{T}_{M,0}$. The rest is clear.

b) Assume that such a generating family F exists. (L, λ) is its own maximalisation. Therefore the homeomorphism \mathbf{q} is an isomorphism.

Then $\mathbf{a}_C = \hat{\mathbf{q}} \circ \mathbf{a}$ (cf. Definition 15.4 and Theorem 15.1) is an isomorphism. We are simultaneously in the special cases Definition 15.4 and Corollary 15.2 of Theorem 15.1. By Lemma 16.8 (L, λ) is smooth, a contradiction. \square

18. COXETER GROUPS AND F-MANIFOLDS

The complex orbit space of an irreducible Coxeter group is equipped with the discriminant, the image of the reflection hyperplanes, and with a certain distinguished vector field (see below), which is unique up to a scalar. Together they induce as in Corollary 11.6 the structure of an F -manifold on the complex orbit space (Theorem 18.1). This follows independently from [Du1][Du2](Lecture 4) and from [Gi2](Theorem 14).

In fact, both give stronger results. Dubrovin established the structure of a Frobenius manifold. This will be discussed in chapter 19. Givental proved that these F -manifolds are distinguished by certain

geometric conditions (Theorem 18.4). With one additional argument we will show that the germs of these F-manifolds and their products are the only germs of simple F-manifolds whose tangent spaces are Frobenius algebras (Theorem 18.3). This complements in a nice way the relation between Coxeter groups and simple hypersurface and boundary singularities.

We will also present simple explicit formulas for these F-manifolds which are new for H_3 and H_4 (Theorem 18.5).

A Coxeter group is a finite group W of linear transformations of the Euclidean space \mathbb{R}^n generated by reflections in hyperplanes. Each Coxeter group is the direct sum of irreducible Coxeter groups. They are ([Co][Bo]) A_n ($n \geq 1$), D_n ($n \geq 4$), E_6 , E_7 , E_8 , B_n , ($n \geq 2$), F_4 , G_2 , H_3 , H_4 , $I_2(m)$ ($m \geq 3$) with $A_2 = I_2(3)$, $B_2 = I_2(4)$, $H_2 := I_2(5)$, $G_2 = I_2(6)$.

The Coxeter group W acts on $\mathbb{C}^n = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ and on $\mathbb{C}[x_1, \dots, x_n]$, where x_1, \dots, x_n are the coordinates on \mathbb{C}^n . The ring $\mathbb{C}[x_1, \dots, x_n]^W$ of invariant polynomials is generated by n algebraically independent homogeneous polynomials P_1, \dots, P_n . Their degrees $d_i := \deg P_i$ are unique (up to ordering). The quotient \mathbb{C}^n/W is isomorphic to \mathbb{C}^n as affine algebraic variety. The \mathbb{C}^* -action and the vector field $\sum_i x_i \frac{\partial}{\partial x_i}$ on the original \mathbb{C}^n induce a \mathbb{C}^* -action and a vector field $\sum d_i t_i \frac{\partial}{\partial t_i}$ on the orbit space $\mathbb{C}^n/W \cong \mathbb{C}^n$. The image in the orbit space of the union of the reflection hyperplanes is the discriminant \mathcal{D} of the Coxeter group.

Suppose for a moment that W is irreducible. Then there is precisely one highest degree, which is called the Coxeter number h . The degrees can be ordered to satisfy

$$d_1 = h > d_2 \geq \dots \geq d_{n-1} > d_n = 2, \quad (18.1)$$

$$d_i + d_{n+1-i} = h + 2. \quad (18.2)$$

The vector field $e := \frac{\partial}{\partial t_1}$ is unique up to a scalar.

Theorem 18.1. *The complex orbit space $M := \mathbb{C}^n/W \cong \mathbb{C}^n$ of an irreducible Coxeter group W carries a unique structure of a massive F-manifold with the unit field $e = \frac{\partial}{\partial t_1}$ and the discriminant \mathcal{D} . The discriminant \mathcal{D} corresponds to the Euler field*

$$E = \frac{1}{h} \sum_{i=1}^n d_i t_i \frac{\partial}{\partial t_i} \quad (18.3)$$

of weight 1.

Proof: The uniqueness follows from Corollary 11.6.

The existence follows from Dubrovin's result ([Du1][Du2] (Lecture 4), cf. Theorem 19.1) or Givental's result [Gi2](Theorem 14) together with Theorem 9.4.

Below in Theorem 18.5 we will follow Givental and reduce it to classical results on the appearance of discriminants in singularity theory ([Bri][Ar2][Ly2][ShO]). \square

Remarks 18.2. i) Corollary 11.6 gives probably the most elementary way how e and \mathcal{D} determine the multiplication on the complex orbit space $M = \mathbb{C}^n/W \cong \mathbb{C}^n$, at least at a generic point: the e -orbit of a generic point $p \in M$ intersects \mathcal{D} transversally in n points. One shifts the tangent spaces of \mathcal{D} at these points with the flow of e to $T_p M$. Then there exists a basis e_1, \dots, e_n of $T_p M$ such that $\sum_{i=1}^n e_i = e$ and such that the hyperplanes $\bigoplus_{i \neq j} \mathbb{C} \cdot e_i$, $j = 1, \dots, n$, are the shifted tangent spaces of \mathcal{D} . The multiplication on $T_p M$ is given by $e_i \circ e_j = \delta_{ij} e_i$.

ii) The unit field $e = \frac{\partial}{\partial t_1}$ is only unique up to a scalar. The flow of the Euler field respects the discriminant \mathcal{D} and maps the unit field e and the multiplication to multiples, because of $\text{Lie}_E(e) = -e$ and $\text{Lie}_E(\circ) = \circ$.

Therefore the isomorphism class of the F-manifold (M, \circ, e, E) is independent of the choice of the scalar.

iii) The complex orbit space of a reducible Coxeter group W is isomorphic to the product of the complex orbit spaces of the irreducible subgroups. The discriminant decomposes as in Remark 11.2 iv). Now any sum of unit fields for the components yields a unit field for \mathbb{C}^n/W . The choices are parametrized by $(\mathbb{C}^*)^{|irr.subgroups|}$. But the resulting F-manifold is unique up to isomorphism. It is the product of F-manifolds for the irreducible subgroups. This F-manifold and its germ at 0 will be denoted by the same combination of letters as the Coxeter group.

Theorem 18.3. *Let $((M, p), \circ, e)$ be a germ of a massive F-manifold.*

$((M, p), \circ, e)$ is simple and $T_p M$ is a Frobenius algebra if and only if $((M, p), \circ, e)$ is isomorphic to the germ at 0 of an F-manifold of a Coxeter group.

This builds on the following result, which is a reformulation with chapter 9 of a theorem of Givental [Gi2](Theorem 14). The Theorems 18.3, 18.4, and 18.5 will be proved below in the opposite order. Some arguments on H_3 and H_4 in the proof of Theorem 18.4 will only be sketched.

Theorem 18.4. (Givental) *a) The F-manifold of an irreducible Coxeter group is simple. The analytic spectrum (L, λ) of its germ at 0 is*

isomorphic to $(\{(x, y) \in \mathbb{C}^2 \mid x^2 = y^r\}, 0) \times (\mathbb{C}^{n-1}, 0)$ with $r = 1$ for A_n, D_n, E_n , $r = 2$ for B_n, F_4 , $r = 3$ for H_3, H_4 and $r = m - 2$ for $I_2(m)$.

b) An irreducible germ of a simple F -manifold with analytic spectrum isomorphic to a product of germs of plane curves is isomorphic to the germ at 0 of an F -manifold of an irreducible Coxeter group.

Finally, we want to present the F -manifolds of the irreducible Coxeter groups explicitly with data as in Corollary 15.2. We will use the notations of Corollary 15.2. The following is a consequence of results in [Bri][Ar2][Ly2][ShO] on the appearance of the discriminants of Coxeter groups in singularity theory.

Theorem 18.5. a) The germs at 0 of the F -manifolds of the Coxeter groups A_n, D_n, E_n, B_n, F_4 are isomorphic to the base spaces of the semiuniversal unfoldings of the corresponding simple hypersurface singularities A_n, D_n, E_n and simple boundary singularities B_n (or C_n) and F_4 .

b) For the F -manifolds $(M, \circ, e) = (\mathbb{C}^n, \circ, e)$ of the irreducible Coxeter groups, a space Z with projection $\pi_Z : Z \rightarrow M$, a subspace $C \subset Z$ and a 1-form α_Z will be given such that the map

$$\mathbf{a}_C : \mathcal{T}_M \rightarrow (\pi_C)_* \mathcal{O}_C, \quad X \mapsto \alpha_Z(\tilde{X})|_C \quad (18.4)$$

is welldefined and an isomorphism of \mathcal{O}_M -algebras. C is isomorphic to the analytic spectrum of (M, \circ, e) . The Euler field is always $E = \frac{1}{h} \sum_{i=1}^n dt_i \frac{\partial}{\partial t_i}$. The discriminant $\mathcal{D} \subset M$ is $\mathcal{D} = \pi_C(\mathbf{a}_C(E)^{-1}(0))$.

i) $A_n, B_n, H_3, I_2(m)$:

$Z = \mathbb{C} \times M = \mathbb{C} \times \mathbb{C}^n$ with coordinates $(x, t) = (x, t_1, \dots, t_n)$,

$\alpha_Z = dt_1 + xdt_2 + \dots + x^{n-1}dt_n$,

$t_2(x, t) := t_2 + 2xt_3 + \dots + (n-1)x^{n-2}t_n$,

$A_n : C = \{(x, t) \in Z \mid x^n - \tilde{t}_2 = 0\}$,

$B_n : C = \{(x, t) \in Z \mid x \cdot (x^{n-1} - \tilde{t}_2) = 0\}$,

$H_3 : C = \{(x, t) \in Z \mid x^3 - \tilde{t}_2^2 = 0\}$,

$I_2(m) : C = \{(x, t) \in Z \mid x^2 - \tilde{t}_2^{m-2} = 0\}$.

ii) D_4, F_4, H_4 :

$Z = \mathbb{C}^2 \times M = \mathbb{C}^2 \times \mathbb{C}^4$ with coordinates $(x, y, t) = (x, y, t_1, \dots, t_4)$,

$\alpha_Z = dt_1 + xdt_2 + ydt_3 + xydt_4$,

$t_2(x, y, t) := t_2 + yt_4$, $t_3(x, y, t) := t_3 + xt_4$,

$D_4 : C = \{(x, y, t) \in Z \mid x^2 + \tilde{t}_2 = 0, y^2 + \tilde{t}_3 = 0\}$,

$F_4 : C = \{(x, y, t) \in Z \mid x^2 + \tilde{t}_2 = 0, y^2 + \tilde{t}_3^2 = 0\}$,

$H_4 : C = \{(x, y, t) \in Z \mid x^2 + \tilde{t}_2 = 0, y^2 + \tilde{t}_3^3 = 0\}$.

iii) D_n, E_6, E_7, E_8 :
 $Z = \mathbb{C}^2 \times M = \mathbb{C}^2 \times \mathbb{C}^n$ with coordinates $(x, y, t) = (x, y, t_1, \dots, t_n)$,
 $F : Z \rightarrow \mathbb{C}$ a semiuniversal unfolding of $F|_{\mathbb{C}^2 \times \{0\}}$,
 $\alpha_Z = \sum_{i=1}^n \frac{\partial F}{\partial t_i} dt_i$ (or $\alpha_Z = dF$),
 $C = \{(x, y, t) \in Z \mid \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0\}$,
 $D_n : F = x^{n-1} + xy^2 + \sum_{i=1}^{[n/2]} x^{i-1} t_i + yt_{[n/2]+1} + \sum_{i=[n/2]+2}^n x^{i-2} t_i$,
 $E_6 : F = x^4 + y^3 + t_1 + xt_2 + yt_3 + x^2 t_4 + xyt_5 + x^2 yt_6$,
 $E_7 : F = x^3 y + y^3 + t_1 + xt_2 + yt_3 + x^2 t_4 + xyt_5 + x^3 t_6 + x^4 t_7$,
 $E_8 : F = x^5 + y^3 + t_1 + xt_2 + yt_3 + x^2 t_4 + xyt_5 + x^3 t_6 + x^2 yt_7 + x^3 yt_8$.

Proof of Theorem 18.5: a) One can choose a semiuniversal unfolding $F = f(x_1, \dots, x_m) + \sum_{i=1}^n m_i t_i$ of the hypersurface or boundary singularity which is weighted homogeneous with positive degrees in all variables and parameters. There is an isomorphism from its base space \mathbb{C}^n to the complex orbit space of the corresponding Coxeter group which respects the discriminant, the Euler field, and the unit field ([Bri][Ar2]). It respects also the F-manifold structure (Corollary 11.6).

b) i) for $I_2(m)$ is Remark 15.5 a). i) for A_n, B_n and ii) for D_4 follow with a), with semiuniversal unfolding as in a) for the singularities $-\frac{1}{n+1}x^{n+1}$ (A_n), $(-\frac{1}{n}x^n + y^2, H = \{x = 0\})$ (B_n), $\frac{1}{3}x^3 + \frac{1}{3}y^3$ (D_4). Also iii) follows with a).

The same procedure gives for the boundary singularity $(\frac{1}{2}y^2 + \frac{1}{3}x^3, H = \{y = 0\})$ (F_4) the data in ii) with critical set

$$C' = \{(x, y, t) \in Z \mid x^2 + \tilde{t}_2 = 0, y^2 + y\tilde{t}_3 = 0\}. \quad (18.5)$$

It is a nontrivial, but solvable exercise to find compatible automorphisms of Z and M which map C' to C and α_Z to α_Z modulo $I_C \Omega_Z^1$. Independently of explicit calculations, the proof of Theorem 18.4 will show that the data (Z, α_Z, C) in ii) correspond to F_4 .

The data in i) for H_3 and in ii) for H_4 can be obtained from results of O.P. Shcherbak ([ShO] pp 162, 163; [Gi2] Proposition 12) (for H_3 one could use instead [Ly2]). The unfoldings

$$F_{H_3} = \int_0^y (u^2 + x)^2 du + t_1 + xt_2 + x^2 t_3 \quad (18.6)$$

of D_6 and

$$F_{H_4} = \int_0^y (u^2 + t_3 + xt_4)^2 du + x^3 + t_1 + xt_2 \quad (18.7)$$

of E_8 have only critical points with even Milnor number and are maximal with this property. Their discriminants are isomorphic to the discriminants of the Coxeter groups H_3 and H_4 . The unfoldings are

generating families in the sense of Definition 15.4 for the F-manifolds of the Coxeter groups H_3 and H_4 . We will determine the data in ii) for H_4 from F_{H_4} ; the case H_3 is similar.

Consider the map

$$\phi : \mathbb{C}^2 \times \mathbb{C}^4 \rightarrow Z = \mathbb{C}^2 \times \mathbb{C}^4, (x, y, t) \mapsto (x, \tilde{y}, t), \quad (18.8)$$

$$\tilde{y}(x, y, t) := \int_0^y 2(u^2 + t_3 + xt_4)du = \frac{2}{3}y^3 + 2(t_3 + xt_4)y, \quad (18.9)$$

and observe

$$\begin{aligned} dF_{H_4} &= (3x^2 + t_2 + \tilde{y}t_4)dx + (y^2 + t_3 + xt_4)^2dy \\ &\quad + dt_1 + xdt_2 + \tilde{y}dt_3 + x\tilde{y}dt_4, \end{aligned} \quad (18.10)$$

$$\frac{9}{4}\tilde{y}^2 + 4(t_3 + xt_4)^3 = (y^2 + t_3 + xt_4)^2(y^2 + 4(t_3 + xt_4)). \quad (18.11)$$

Therefore

$$\phi^*(\alpha_Z) = dF - \frac{\partial F}{\partial x}dx - \frac{\partial F}{\partial y}dy \quad (18.12)$$

and the image under ϕ of the reduced critical set C_F of F_{H_4} is

$$\begin{aligned} \phi(C_F) = \{(x, y, t) \in Z \mid & 3x^2 + t_2 + yt_4 = 0, \\ & \frac{9}{16}y^2 + t_3 + xt_4 = 0\}. \end{aligned} \quad (18.13)$$

An automorphism

$$Z \rightarrow Z, (x, y, t_1, t_2, t_3, t_4) \mapsto (r^{-1}x, s^{-1}y, t_1, rt_2, st_3, rst_4) \quad (18.14)$$

for suitable $r, s \in \mathbb{C}^*$ maps $\phi(C_F)$ to C and respects π_Z and α_Z , together with the induced automorphism $M \rightarrow M$. \square

Sketch of the proof of Theorem 18.4: a) Consider the data in Theorem 18.5 b). The Euler field on $M = \mathbb{C}^n$ is $E = \frac{1}{h} \sum_i d_i t_i \frac{\partial}{\partial t_i}$. The coefficients of the Lyashko-Looijenga map $\Lambda : M \rightarrow \mathbb{C}^n$ are up to a sign the symmetric polynomials in the eigenvalues of $E \circ$. Because of $\text{Lie}_E(E \circ) = E \circ$, the coefficient Λ_i is weighted homogeneous of degree i with respect to the weights $(\frac{d_1}{h}, \dots, \frac{d_n}{h})$ for (t_1, \dots, t_n) . The Lyashko-Looijenga map is a branched covering of degree

$$n! \cdot \left(\prod \frac{d_i}{h}\right)^{-1} = n!h^n \cdot |W|^{-1} \quad (18.15)$$

and (M, \circ, e) is simple (Corollary 14.4 b)). The analytic spectrum is isomorphic to C .

b) The dimension $\dim(M_k, p)$ of an irreducible germ in the decomposition $(M, p) = \prod_{k=1}^l (M_k, p)$ of a germ of a massive F-manifold is

equal to the intersection multiplicity of T_p^*M with the corresponding germ (L, λ_k) of the analytic spectrum L . This number will be called the intersection multiplicity $IM(\lambda_k)$.

$(S_\mu(q), q)$ denotes for any $q \in M$ the μ -constant stratum through q (chapter 14), and $l(q)$ the number of irreducible components of (M, q) . For any subvariety $S \subset L$ we have the estimates

$$\max(l(q) | q \in \pi(S)) \leq n + 1 - \min(IM(\sigma) | \sigma \in S) , \quad (18.16)$$

$$\max(\dim(S_\mu(q), q) \mid q \in S) \geq \dim S , \quad (18.17)$$

$$\begin{aligned} & \max(\text{mod}_\mu(M, q) \mid q \in S) \\ & + n + 1 - \min(IM(\sigma) \mid \sigma \in S) \geq \dim S . \end{aligned} \quad (18.18)$$

Therefore, if M is simple then

$$\min(IM(\sigma) \mid \sigma \in S) \leq n + 1 - \dim S \quad (18.19)$$

for any subvariety $S \subset L$.

Now suppose that $((M, p), \circ, e)$ is an irreducible germ of a simple F-manifold and that $\phi : (M, p) \rightarrow \prod_{i=1}^n (C_i, 0)$ is an isomorphism to a product of germs of plane curves (they are necessarily plane because of Proposition 7.4).

If at least two curve germs were not smooth, e.g. $(C_{n-1}, 0)$ and $(C_n, 0)$, then the intersection multiplicities $IM(p)$ for points p in $S_1 := \phi^{-1}(\prod_{i=1}^{n-2} (C_i, 0) \times \{0\})$ were at least 4; but $\dim(S_1, p) = n - 2$, a contradiction to (18.19). So, at most one curve, e.g. $(C_n, 0)$, is not smooth.

The irreducible germs of F-manifolds which correspond to generic points of $\pi(S_2)$ for $S_2 := \phi^{-1}(\prod_{i=1}^{n-1} (C_i, 0) \times \{0\})$ are at most 2-dimensional because of (18.19). Therefore

$$(C_n, 0) \cong (\{(x, y) \in \mathbb{C}^2 \mid x^2 = y^r\}, 0) \quad (18.20)$$

for some $r \in \mathbb{N}$. If $r \geq 4$ and $n \geq 3$ then the set of possible intersection multiplicities for points in S_2 has a gap at 3 and a subvariety $S_3 \subset S_2$ exists with $\dim S_3 = n - 2$ and $\min(IM(\sigma) \mid \sigma \in S_3) \geq 4$ [Gi2](p 3266), a contradiction to (18.19). Therefore $r \in \{1, 2, 3\}$ or $n \leq 2$.

If $r \in \{1, 2\}$ then (M, p) is the base space of a semiuniversal unfolding of a hypersurface singularity ($r = 1$, Theorem 16.7, [AGV](19.)) or boundary singularity ($r = 2$, Theorem 17.7, [NN]). Simplicity of their F-manifolds corresponds to simplicity of the singularities. The simple hypersurface singularities are A_n, D_n, E_6, E_7, E_8 [AGV]. The simple boundary singularities are B_n, C_n , and F_4 [Ar2][AGV]. B_n and C_n are dual boundary singularities and have isomorphic discriminants and F-manifolds.

The details of the case $r = 3$ ([Gi2] pp 3269–3271) are difficult and will not be given here. In that case the set of possible intersection multiplicities for points in S_2 has a gap at 5. If $n \geq 6$ then a subvariety $S_4 \subset S_2$ exists with $\dim S_4 = n - 4$ and $\min(IM(\sigma) \mid \sigma \in S_4) \geq 6$, a contradiction to (18.19). The case $r = 3$, $n = 3$ corresponds to H_3 , the case $r = 3$, $n = 4$ corresponds to H_4 . \square

Proof of Theorem 18.3: It is sufficient to consider an irreducible germ $((M, p), \circ, e)$. If it corresponds to a Coxeter group then it is simple (Theorem 18.4 a)) and $T_p M$ is a Frobenius algebra (Theorem 18.5 b)).

Suppose that (M, p) is simple and that $T_p M$ is a Frobenius algebra. We will show by induction on the dimension $n = \dim M$ that the analytic spectrum (L, λ) is isomorphic to $(\{(x, y) \in \mathbb{C}^2 \mid x^2 = y^r\}, 0) \times (\mathbb{C}^{n-1}, 0)$ for some $r \in \mathbb{N}$.

This is clear for $n = 2$. Suppose that $n \geq 3$. The maximal ideal of $T_p M$ is called \mathbf{m} . The socle $\text{Ann}_{T_p M}(\mathbf{m})$ of the Gorenstein ring $T_p M$ has dimension 1, therefore $\text{Ann}_{T_p M}(\mathbf{m}) \subsetneq \mathbf{m}$ and $\mathbf{m}^2 \neq 0$. In the equations for the analytic spectrum $(L, \lambda) \subset T_p^* M$ one can eliminate fiber coordinates which correspond to $\mathbf{m}^2 \subset T_p M$: the embedding dimension of (L, λ) is

$$\text{embdim}(L, \lambda) \leq n + \dim \frac{\mathbf{m}}{\mathbf{m}^2} \leq 2n - 2 \quad (18.21)$$

(Lemma 15.3). Then $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (L'', \lambda'')$ (Proposition 7.4). There exists $\lambda_2 \in L$ close to λ such that $(L, \lambda_2) \cong (L, \lambda)$ and $\pi(\lambda_2)$ is not in the e -orbit of p .

Now for all q near p , but outside of the e -orbit of p the germ (M, q) is reducible because of $\text{mod}_\mu(M, p) = 0$. For all q near p the germ (M, q) is simple and $T_q M$ is a Frobenius algebra (Lemma 1.2).

One can apply the induction hypothesis to the irreducible component of $(M, \pi(\lambda_2))$ which corresponds to λ_2 . Its analytic spectrum (L', λ') is isomorphic to a product of a smooth germ and a curve as above. Now $(L, \lambda) \cong (L, \lambda_2) \cong (\mathbb{C}^{n-\dim L'}, 0) \times (L', \lambda')$. One applies Theorem 18.4 b). \square

19. COXETER GROUPS AND FROBENIUS MANIFOLDS

K. Saito [SaK2] introduced a flat metric on the complex orbit space of an irreducible Coxeter group. Dubrovin [Du1][Du2](Lecture 4) showed that this metric and the multiplication and the Euler field from Theorem 18.1 together yield the structure of a massive Frobenius manifold on the complex orbit space (Theorem 19.1).

The Euler field has positive degrees. Dubrovin [Du1][Du2](p 268) conjectured that these Frobenius manifolds and products of them are the only massive Frobenius manifolds with an Euler field with positive degrees.

We will prove this conjecture (Theorem 19.3). Theorem 18.3, which builds on Givental's result (Theorem 18.4, [Gi2](Theorem 14)), will be crucial.

We use the same notations as in chapter 18. A metric on a complex manifold is a nondegenerate complex bilinear form on the tangent bundle. The flat standard metric on \mathbb{C}^n is invariant with respect to the Coxeter group W and induces a flat metric \check{g} on $M - \mathcal{D}$. Dubrovin proved the following with differential geometric tools [Du1][Du2](Lecture 4 and pp 191, 195).

Theorem 19.1. (Dubrovin) *Let W be an irreducible Coxeter group with complex orbit space $M = \mathbb{C}^n/W$, Euler field E , a unit field e , and a multiplication \circ on M as in Theorem 18.1.*

The metric g on $M - \mathcal{D}$ with

$$g(X, Y) := \check{g}(E \circ X, Y) \quad (19.1)$$

for any (local) vector fields X and Y extends to a flat metric on M and coincides with K. Saito's flat metric. (M, \circ, e, E, g) is a Frobenius manifold. The Euler field satisfies

$$\text{Lie}_E(g) = (1 + \frac{2}{h})g . \quad (19.2)$$

There exists a basis of flat coordinates z_1, \dots, z_n on M with $z_i(0) = 0$ and $e = \frac{\partial}{\partial z_1}$ and

$$E = \frac{1}{h} \sum d_i \cdot z_i \frac{\partial}{\partial z_i} . \quad (19.3)$$

Remarks 19.2. i) K. Saito (and also Dubrovin) introduced the flat metric g in a way different from formula (19.2): The metrics \check{g} and g on $M - \mathcal{D}$ induce two isomorphisms $T(M - \mathcal{D}) \rightarrow T^*(M - \mathcal{D})$. One lifts \check{g} and g with the respective isomorphism to metrics \check{g}^* and g^* on the cotangent bundle $T^*(M - \mathcal{D})$. Then

$$g^* = \text{Lie}_e(\check{g}^*) \quad (19.4)$$

([Du2] pp 191, 195). (Here \check{g}^* and g^* are considered as $(0, 2)$ -tensors.) K. Saito introduced g with formula (19.4).

ii) Closely related to (19.2) and (19.4) is ([Du2] pp 191, 270)

$$\sum_{i=1}^n \frac{\partial Q_1}{\partial x_i} \cdot \frac{\partial Q_2}{\partial x_i} = \check{g}^*(dQ_1, dQ_2) = i_E(dQ_1 \circ dQ_2) . \quad (19.5)$$

Here $Q_1, Q_2 \in \mathbb{C}[x_1, \dots, x_n]^W$ are W -invariant polynomials; dQ_1 and dQ_2 are interpreted as sections in T^*M ; the multiplication \circ is lifted to T^*M with the isomorphism $TM \rightarrow T^*M$ induced by g ; i_E is the contraction of a 1-form with E .

The first equality is trivial. (19.5) is related to Arnold's convolution of invariants ([Ar3][Gi1]).

iii) A Frobenius manifold as in Theorem 19.1 for an irreducible Coxeter group is not unique as the unit field and the multiplication are not unique. Contrary to the F-manifold, it is not even unique up to isomorphism. There is one complex parameter between (M, \circ, e) and (M, g) to be chosen: $(M, \circ, e, E, c \cdot g)$ respectively $(M, c \cdot \circ, c^{-1} \cdot e, E, g)$ is a Frobenius manifold for any $c \in \mathbb{C}^*$.

iv) We consider only Frobenius manifolds with an Euler field which is normalized by $\text{Lie}_E(\circ) = 1 \cdot \circ$ (compare Remark 5.5 c)). The product $\prod M_i$ of Frobenius manifolds $(M_i, \circ_i, e_i, E_i, g_i)$ carries also the structure of a Frobenius manifold if $\text{Lie}_{E_i}(g_i) = D \cdot g_i$ holds with the same number $D \in \mathbb{C}$ for all i . This follows from Proposition 4.1, Theorem 5.3 and Remark 5.5 c) (compare also [Du2](p 136)).

v) Especially, the complex orbit space \mathbb{C}^n/W of a reducible Coxeter group can be provided with the structure of a Frobenius manifold if the irreducible Coxeter subgroups have the same Coxeter number.

The Frobenius manifold is not unique. The different choices are parametrized by $(\mathbb{C}^*)^{|irr.subgroups|}$ in the obvious way (cf. the Remarks 18.2 iii) and 19.2 iii)).

The *spectrum* of a Frobenius manifold (M, \circ, e, E, g) is defined as follows. ∇ is the Levi-Civita connection of the metric g . The operator $\nabla E : \mathcal{T}_M \rightarrow \mathcal{T}_M$, $X \mapsto \nabla_X E$, acts on the space of flat fields ([Du2](p 132), [Man](p 24)) and coincides there with $-\text{ad } E$. The set of its eigenvalues $\{w_1, \dots, w_n\}$ is the spectrum ([Man]). If $-\text{ad } E$ acts semisimple on the space of flat fields then there exists locally a basis of flat coordinates z_1, \dots, z_n with

$$E = \sum_i (w_i z_i + r_i) \frac{\partial}{\partial z_i} \quad (19.6)$$

for some $r_i \in \mathbb{C}$.

The following was conjectured by Dubrovin ([Du1][Du2](p 268)).

Theorem 19.3. *Let $((M, p), \circ, e, E, g)$ be the germ of a Frobenius manifold with the following properties:
 generically semisimple multiplication;
 $\text{Lie}_E(\circ) = 1 \cdot \circ$ and $\text{Lie}_E(g) = D \cdot g$;*

$$E = \sum w_i z_i \frac{\partial}{\partial z_i} \quad (19.7)$$

*for a basis of flat coordinates z_i with $z_i(p) = 0$;
 positive spectrum (w_1, \dots, w_n) , that is, $w_i > 0$ for all i .*

Then (M, p) decomposes uniquely into a product of germs at 0 of Frobenius manifolds for certain irreducible Coxeter groups. The Coxeter groups have all the same Coxeter number $h = \frac{2}{D-1}$.

Proof: As in the proof of Theorem 18.4 a), the hypotheses on the Euler field show that the Lyashko-Looijenga map $\Lambda : (M_k, p) \rightarrow \mathbb{C}^n$ is finite and that the F-manifold (M, \circ, e) is simple. One applies Theorem 18.3 and Theorem 19.4. \square

Theorem 19.4. *Let $((M, p), \circ, e, E, g)$ be the germ of a Frobenius manifold such that $((M, p), \circ, e, E)$ is isomorphic to the germ at 0 of the F-manifold of a Coxeter group with the standard Euler field.*

Then the irreducible Coxeter subgroups have the same Coxeter number and $((M, p), \circ, e, E, g)$ is isomorphic to a product of germs at 0 of Frobenius manifolds for these Coxeter groups.

Proof: First we fix notations. W is a Coxeter group which acts on $V = \mathbb{C}^n$ and respects the standard bilinear form. The decomposition of W into l irreducible Coxeter groups W_1, \dots, W_l corresponds to an orthogonal decomposition $V = \bigoplus_{k=1}^l V_k$. The choice of n algebraically independent homogeneous polynomials $P_1, \dots, P_n \in \mathbb{C}[x_1, \dots, x_n]^W$ identifies the quotient $M = V/W$ with \mathbb{C}^n . The quotient map $\psi : V \rightarrow M$ decomposes into a product of quotient maps $\psi_k : V_k \rightarrow V_k/W_k = M_k$. The F-manifold $M \cong \prod_{k=1}^l M_k$ is the product of the F-manifolds M_k .

Setting $\varepsilon := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ on V and $\varepsilon_k := \varepsilon|_{V_k}$, the standard Euler field E_k on M_k is $E_k = \frac{1}{h_k} d\psi_k(\varepsilon_k)$. Here h_k is the Coxeter number of W_k . The Euler field on M is

$$E = \sum_{k=1}^l E_k = \sum_{i=1}^n w_i t_i \frac{\partial}{\partial t_i}, \quad (19.8)$$

$\{w_1, \dots, w_n\}$ is the union of the invariant degrees of W_k , divided by h_k .

Now suppose that g is a flat metric on the germ $(M, 0)$ such that $((M, 0), \circ, e, E, g)$ is a germ of a Frobenius manifold with $\text{Lie}_E(g) = D \cdot g$. Consider a system of flat coordinates z_1, \dots, z_n of $(M, 0)$ with

$z_i(0) = 0$. The space $\bigoplus_{i=1}^n \mathbb{C} \cdot \frac{\partial}{\partial z_i}$ of flat fields is invariant with respect to $\text{ad } E$ ([Du2] p 132, [Man] p 24) and the space of affine linear functions $\mathbb{C} \cdot 1 \oplus \bigoplus_{i=1}^n \mathbb{C} \cdot z_i \subset \mathcal{O}_{M,0}$ is invariant with respect to E . Because E vanishes at 0 even the subspace $\bigoplus_{i=1}^n \mathbb{C} \cdot z_i$ is invariant with respect to E .

The weights w_1, \dots, w_n of E are positive. Therefore the coordinates z_1, \dots, z_n can be chosen to be weighted homogeneous polynomials in $\mathbb{C}[t_1, \dots, t_n]$ of degree w_1, \dots, w_n . Thus the spectrum of the Frobenius manifold is $\{w_1, \dots, w_n\}$. It is symmetric with respect to $\frac{D}{2}$, because of $\text{Lie}_E(g) = D \cdot g$; hence $1 + \frac{2}{h_k} = D$ for all $k = 1, \dots, l$. The Coxeter numbers are all equal, $h := h_1 = \dots = h_l$, the Euler field E is

$$E = \frac{1}{h} d\psi(\varepsilon) . \quad (19.9)$$

It rests to show that g is induced as in Theorem 19.1 from a metric on V which is the orthogonal sum of multiples of the standard metrics on the subspaces V_k .

The operator $\mathcal{U} = E \circ : \mathcal{T}_M \rightarrow \mathcal{T}_M$ is invertible on $M - \mathcal{D}$. The metric \check{g} on $M - \mathcal{D}$ with

$$\check{g}(X, Y) := g(\mathcal{U}^{-1}(X), Y) \quad (19.10)$$

is flat ([Du2] pp 191, 194; [Man]). It lifts to a flat metric \tilde{g} on $V - \psi^{-1}(\mathcal{D})$. We claim that \tilde{g} extends to a flat metric on the union $\psi^{-1}(\mathcal{D})$ of the reflection hyperplanes.

It is sufficient to consider a generic point p in one reflection hyperplane. Then the e -orbit of $\psi(p)$ intersects \mathcal{D} in n points; there exist canonical coordinates u_1, \dots, u_n in a neighborhood of $\psi(p)$ with $e_i \circ e_j = \delta_{ij} e_i$, $g(e_i, e_j) = 0$ for $i \neq j$,

$$E = u_1 e_1 + \sum_{i=2}^n (u_i + r_i) e_i \quad \text{for some } r_i \in \mathbb{C}^* ,$$

$$(\mathcal{D}, \psi(p)) \cong (\{u \mid u_1 = 0\}, 0) .$$

The map germ $\psi : (V, p) \rightarrow (M, \psi(p))$ is a twofold covering, branched along $(\mathcal{D}, \psi(p))$, and is given by $(\tilde{u}_1, \dots, \tilde{u}_n) \mapsto (\tilde{u}_1^2, \tilde{u}_2, \dots, \tilde{u}_n) = (u_1, \dots, u_n)$ for some suitable local coordinates $\tilde{u}_1, \dots, \tilde{u}_n$ on (V, p) . Then

$$\begin{aligned} \tilde{g}\left(\frac{\partial}{\partial \tilde{u}_1}, \frac{\partial}{\partial \tilde{u}_1}\right) &= \check{g}(4u_1 e_1, e_1) = \check{g}(4E \circ e_1, e_1) = 4g(e_1, e_1) , \\ \tilde{g}\left(\frac{\partial}{\partial \tilde{u}_i}, \frac{\partial}{\partial \tilde{u}_i}\right) &= \check{g}(e_i, e_i) = \frac{1}{u_i + r_i} g(e_i, e_i) \quad \text{for } i \geq 2 , \\ \tilde{g}\left(\frac{\partial}{\partial \tilde{u}_i}, \frac{\partial}{\partial \tilde{u}_j}\right) &= 0 \quad \text{for } i \neq j . \end{aligned}$$

So \tilde{g} extends to a nondegenerate (and then flat) metric on V .

The Coxeter group W acts as group of isometries with respect to \tilde{g} . It rests to show that the vector space structure on V which is induced by \tilde{g} (and $0 \in V$) coincides with the old one. Then \tilde{g} is an orthogonal sum of multiples of the standard metrics on the subspaces V_k , because each W -invariant quadratic form is a sum of W_k -invariant quadratic forms on the subspaces V_k and they are unique up to scalars.

Let $\tilde{\varepsilon}$ be the vector field on V which corresponds to the \mathbb{C}^* -action of the vector space structure induced by \tilde{g} . Then $\text{Lie}_{\tilde{\varepsilon}} \tilde{g} = 2 \cdot \tilde{g}$. Because of $\text{Lie}_E(\mathcal{U}) = \mathcal{U}$, $\text{Lie}_E(g) = (1 + \frac{2}{h})g$, and $E = \frac{1}{h}d\psi(\varepsilon)$ we have also $\text{Lie}_{\varepsilon} \tilde{g} = 2 \cdot \tilde{g}$ for the vector field ε , which corresponds to the \mathbb{C}^* -action of the old vector space structure. The difference $\varepsilon - \tilde{\varepsilon}$ satisfies $\text{Lie}_{\varepsilon - \tilde{\varepsilon}}(\tilde{g}) = 0$ and is a generator of a 1-parameter group of isometries. As it is also tangent to the union of reflection hyperplanes, it vanishes. The vector field $\varepsilon = \tilde{\varepsilon}$ determines a unique space of linear functions on V and a unique vector space structure. \square

20. 3-DIMENSIONAL AND OTHER F-MANIFOLDS

The F-manifolds in chapters 16 – 18 were special in several aspects: the analytic spectrum was weighted homogeneous and a complete intersection. Therefore always an Euler field of weight 1 existed, and the tangent spaces were Frobenius algebras. Furthermore, the stratum of points with irreducible germs of dimension ≥ 3 had codimension 2.

Here we want to present examples with different properties. A partial classification of 3-dimensional germs of massive F-manifolds will show that already in dimension 3 most germs are not simple and do not even have an Euler field of weight 1. Examples of germs (M, p) of simple F-manifolds such that $T_p M$ is not a Frobenius algebra will complement Theorem 18.3.

First, a construction which is behind the formulas for A_n , B_n , H_3 , $I_2(m)$ in Theorem 18.5 b)i) provides many other examples.

Proposition 20.1. *Fix the following data: $(M, 0) = (\mathbb{C}^n, 0)$, $(Z, 0) = (\mathbb{C}, 0) \times (M, 0)$ with coordinates $(x, t) = (x, t_1, \dots, t_n)$, the projection $\pi_Z : (Z, 0) \rightarrow (M, 0)$, the 1-form $\alpha_Z := dt_1 + xdt_2 + \dots + x^{n-1}dt_n$ on Z , the function $t_2(x, t) := t_2 + 2xt_3 + \dots + (n-1)x^{n-2}t_n$, an isolated plane curve singularity (or a smooth germ) $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ with $f(x, 0) = x^n \cdot \text{unit} \in \mathbb{C}\{x\}$, the subvariety $C := \{(x, t) \in Z \mid f(x, \tilde{t}_2) = 0\} \subset Z$.*

a) The map

$$\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}, \quad X \mapsto \alpha_Z(\tilde{X})|_C \quad (20.1)$$

(\tilde{X} is a lift of X to Z) is welldefined and an isomorphism of $\mathcal{O}_{M,0}$ -modules. $(M, 0)$ with the induced multiplication on $\mathcal{T}_{M,0}$ is an irreducible germ of a massive F -manifold. Its analytic spectrum is isomorphic to $(C, 0) \cong (\mathbb{C}^{n-1}, 0) \times (f^{-1}(0), 0)$. For each $t \in M$ the tangent space $T_t M$ is isomorphic to a product of algebras $\mathbb{C}\{x\}/(x^k)$ and is a Frobenius algebra.

b) An Euler field of weight 1 exists on (M, p) iff the curve singularity $f(x, y)$ is weighted homogeneous.

c) Suppose that $\text{mult } f = n$. Then the caustic is $\mathcal{K} = \{t \in M \mid t_2 = 0\}$. The germ (M, t) is irreducible for all $t \in \mathcal{K}$, so the caustic is equal to the μ -constant stratum of $(M, 0)$. The modality is $\text{mod}_\mu(M, 0) = n - 2$ (the maximal possible).

Proof: a) α_Z is exact on C_{reg} because of $d\alpha_Z = dx d\tilde{t}_2$. One can apply Corollary 15.2.

b) Corollary 7.5 b).

c) For $t_2 = 0$ fixed we have $f(x, \tilde{t}_2) = x^n \cdot \text{unit} \in \mathbb{C}\{x, t_3, \dots, t_n\}$. Thus the projection $\pi_C : C \rightarrow M$ is a branched covering of degree n , with $\pi_C^{-1}(\{t \mid t_2 = 0\}) = \{0\} \times \{t \mid t_2 = 0\}$ and unbranched outside of $\{t \mid t_2 = 0\}$. The analytic spectrum is isomorphic to C . \square

Remarks 20.2. i) The function \tilde{t}_2 is part of a coordinate system on T^*M for a different Lagrange fibration: the coordinates

$$\begin{aligned} \tilde{y}_1 &= y_1, \quad \tilde{y}_2 = y_2, \quad \tilde{y}_i = y_i - y_2^{i-1} \quad \text{for } i \geq 3, \\ \tilde{t}_1 &= t_1, \quad \tilde{t}_2 = t_2 + 2y_2 t_3 + \dots + (n-1)y_2^{n-2} t_n, \quad \tilde{t}_i = t_i \quad \text{for } i \geq 3 \end{aligned} \quad (20.2)$$

satisfy

$$\sum_{i=1}^n d\tilde{y}_i d\tilde{t}_i = \sum_{i=1}^n dy_i dt_i = d\alpha. \quad (20.3)$$

The analytic spectrum of an F -manifold as in Proposition 20.1 is

$$L = \{(y, t) \in T^*M \mid \tilde{y}_1 = 1, f(\tilde{y}_2, \tilde{t}_2) = 0, \tilde{y}_i = 0 \text{ for } i \geq 3\}. \quad (20.4)$$

It is a product of Lagrange curves.

ii) Another different Lagrange fibration is behind the formulas for D_4, F_4, H_4 in Theorem 18.5 b)ii). There are many possibilities to generalize the construction of examples above.

In dimension 3, there exist up to isomorphism only two irreducible commutative and associative algebras,

$$Q^{(1)} := \mathbb{C}\{x\}/(x^3) \quad \text{and} \quad (20.5)$$

$$Q^{(2)} := \mathbb{C}\{x, y\}/(x^2, xy, y^2) . \quad (20.6)$$

$Q^{(1)}$ is a Frobenius algebra, $Q^{(2)}$ not.

Theorem 20.3. *Let (M, p) be an irreducible germ of a 3-dimensional massive F -manifold with analytic spectrum $(L, \lambda) \subset T^*M$.*

a) *Suppose $T_p M \cong Q^{(1)}$. Then (L, λ) has embedding dimension 3 or 4 and $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (C', 0)$ for a plane curve $(C', 0) \subset (\mathbb{C}^2, 0)$ with $\text{mult}(C', 0) \leq 3$. An Euler field of weight 1 exists iff $(C', 0)$ is weighted homogeneous.*

b) *Suppose $T_p M \cong Q^{(1)}$ and $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (C', 0)$ with $\text{mult}(C', 0) < 3$. Then $((M, p), \circ, e)$ is one of the germs A_3, B_3, H_3 .*

c) *Suppose $T_p M \cong Q^{(1)}$ and $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (C', 0)$ with $\text{mult}(C', 0) = 3$. Then the caustic \mathcal{K} is a smooth surface and coincides with the μ -constant stratum; that means, $T_q M \cong Q^{(1)}$ for each $q \in \mathcal{K}$. The modality is $\text{mod}_\mu(M, p) = 1$ (the maximal possible).*

d) *Suppose $T_p M \cong Q^{(2)}$. Then (L, λ) has embedding dimension 5 and $(L, \lambda) \cong (\mathbb{C}, 0) \times (L^{(r)}, 0)$. Here $(L^{(r)}, 0)$ is a Lagrange surface with embedding dimension 4. $\mathcal{O}_{L^{(r)}, 0}$ is a Cohen–Macaulay ring, but not a Gorenstein ring.*

Proof: a) One chooses coordinates (t_1, t_2, t_3) for (M, p) (as in the proof of Lemma 15.3) with $e = \frac{\partial}{\partial t_1}$ and

$$\mathbb{C} \cdot \left[\frac{\partial}{\partial t_2} \right] + \mathbb{C} \cdot \left[\frac{\partial}{\partial t_3} \right] = \mathbf{m} \subset T_p M , \quad (20.7)$$

$$\mathbb{C} \cdot \left[\frac{\partial}{\partial t_3} \right] = \mathbf{m}^2 \subset T_p M . \quad (20.8)$$

The dual coordinates on (T^*M, T_p^*M) are y_1, \dots, y_n . There exist functions $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{C}\{t_2, t_3\}$ with

$$\begin{aligned} L &= \{(y, t) \mid y_1 = 1, y_3 = b_2 y_2^2 + b_1 y_2 + b_0, \\ &\quad y_2^3 = a_2 y_2^2 + a_1 y_2 + a_0\} . \end{aligned} \quad (20.9)$$

The Hamilton fields of the smooth functions $y_1 - 1$ and $y_3 - \sum_{i=0}^2 b_i y_2^i$ are $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_3} + \dots$ in T^*M . They are tangent to L . Therefor $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (C', 0)$ with $(C', 0) \cong (L, \lambda) \cap T_p^*M$ (cf. Proposition 7.4). The statement on the Euler field is contained in Corollary 7.5 b).

b) We have $\text{mult}(C', 0) \leq 2$; and the intersection multiplicity of $(C', 0)$ with a suitable smooth curve is 3. So, $(C', 0)$ is either smooth

or a double point or a cusp. In the first two cases, one can apply the correspondence between F-manifolds and hypersurface or boundary singularities (Theorem 16.7 and Theorem 17.7) and the fact that A_3 , B_3 , and C_3 are the only hypersurface or boundary singularities with Milnor number 3.

Suppose $(C', 0)$ is a cusp and $((M, p), \circ, e)$ is not H_3 . Then it is not simple because of Theorem 18.3. The μ -constant stratum $S_\mu = \{q \in M \mid T_q M \cong Q^{(1)}\}$ is more than the e -orbit of p . It can only be the image in M of the surface of cusp points of L , because at other points q close to p the germ (M, q) is A_1^3 or $A_1 \times A_2$.

So at each cusp point λ' of L the intersection multiplicity of $T_{\pi(\lambda')}^* M$ and (L, λ') is 3. This property is not preserved by small changes of the Lagrange fibration (e.g. as in Remark 20.2 i)). But Givental proved that a versal Lagrange map is stable with respect to small changes of the Lagrange fibration ([Gi2] p 3251, Theorem 3 and its proof). This together with chapter 9 yields a contradiction.

c) In this case, at each point λ' (close to λ) of the surface of singular points of L the intersection multiplicity of $T_{\pi(\lambda')}^* M$ and (L, λ') is 3 and the map germ $\pi : (L, \lambda') \rightarrow (M, \pi(\lambda'))$ is a branched covering of degree 3. This implies all the statements.

d) If the embedding dimension of (L, λ) is ≤ 4 then $(L, \lambda) \cong (\mathbb{C}^2, 0) \times (C', 0)$ for some plane curve $(C', 0)$ by Proposition 7.4. Then (L, λ) is a complete intersection and the tangent spaces $T_p M$ are Frobenius algebras.

So if $T_p M \cong Q^{(2)}$ then the embedding dimension of (L, λ) is 5. The ring $\mathcal{O}_{L^{(r)}, 0}$ is a Cohen–Macaulay ring because the projection $L^{(r)} \rightarrow M^{(r)}$ of the restricted Lagrange map is finite and flat. It is not a Gorenstein ring because $T_p M$ is not a Gorenstein ring. \square

The next result provides a complete classification and normal forms for those irreducible germs (M, p) of 3-dimensional massive F-manifolds which satisfy $T_p M \cong Q^{(1)}$ and whose analytic spectrum consists of 3 components. Part a) gives an explicit construction of all those F-manifolds.

Theorem 20.4. *a) Choose two discrete parameters $p_2, p_3 \in \mathbb{N}$ with $p_2 \geq p_3 \geq 2$ and choose $p_3 - 1$ holomorphic parameters $(g_0, g_1, \dots, g_{p_3-2}) \in \mathbb{C}^* \times \mathbb{C}^{p_3-2}$ with $g_0 \neq 1$ if $p_2 = p_3$.*

*Define $(M, 0) := (\mathbb{C}^3, 0)$,
 $(Z, 0) := (\mathbb{C}, 0) \times (M, 0)$ with coordinates $(x, t_1, t_2, t_3) = (x, t)$,
 $g := \sum_{i=0}^{p_3-2} g_i t_2^i + t_2^{p_3-2} \cdot t_3$,
 $f_1 := 0$, $f_2 := t_2^{p_2}$, $f_3 := t_2^{p_3} \cdot g$,*

$C := \bigcup_{i=1}^3 C_i := \bigcup_{i=1}^3 \{(x, t) \in Z \mid x = \frac{\partial f_i}{\partial t_2}\} \subset Z,$
 $b_2(t_2, t_3) := (p_3 \cdot g + t_2 \cdot \frac{\partial g}{\partial t_2})^{-1} \cdot (p_3 \cdot g + t_2 \cdot \frac{\partial g}{\partial t_2} - p_2 \cdot t_2^{p_2-p_3})^{-1}$
 $(b_2 \text{ is a unit in } \mathbb{C}\{t_2, t_3\}),$
 $b_1(t_2, t_3) := -b_2 \cdot p_2 \cdot t_2^{p_2-1},$
 $\text{the 1-form } \alpha_Z := dt_1 + xdt_2 + (b_2x^2 + b_1x)dt_3 \quad \text{on } Z.$

i) Then

$$\frac{\partial f_i}{\partial t_3} = b_2 \left(\frac{\partial f_i}{\partial t_2} \right)^2 + b_1 \frac{\partial f_i}{\partial t_2} \quad \text{for } i = 1, 2, 3, \quad (20.10)$$

$$\alpha_Z|_{C_i} = d(t_1 + f_i)|_{C_i} \quad \text{for } i = 1, 2, 3. \quad (20.11)$$

ii) The map

$$\mathbf{a}_C : \mathcal{T}_{M,0} \rightarrow \mathcal{O}_{C,0}, \quad X \mapsto \alpha_Z(\tilde{X})|_C \quad (20.12)$$

(\tilde{X} is a lift of X to Z) is welldefined and an isomorphism of $\mathcal{O}_{M,0}$ -modules. $(M, 0)$ with the induced multiplication on $\mathcal{T}_{M,0}$ is an irreducible germ of a massive F -manifold with $T_0M \cong Q^{(1)}$. Its analytic spectrum is isomorphic to $(C, 0) \cong (\mathbb{C}^2, 0) \times (C', 0)$, with

$$(C', 0) = (C, 0) \cap (\{(x, t) \mid t_1 = t_3 = 0\}, 0). \quad (20.13)$$

iii) The caustic is $\mathcal{K} = \{t \in M \mid t_2 = 0\}$ and coincides with the bifurcation diagram \mathcal{B} and with the μ -constant stratum.

iv) The functions $t_1 + f_i|_{C_i}$, $i = 1, 2, 3$, combine to a function $F : C \rightarrow \mathbb{C}$ which is continuous on C and holomorphic on $C_{\text{reg}} = C \cap \{(x, t) \mid t_2 \neq 0\}$. The Euler field E on $M - \mathcal{K}$ with $\mathbf{a}_C|_{M-\mathcal{K}}(E) = F|_{M-\mathcal{K}}$ is

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{p_2} t_2 \frac{\partial}{\partial t_2} + \frac{1}{p_2 t_2^{p_3-2}} \left((p_2 - p_3)g - t_2 \frac{\partial g}{\partial t_2} \right) \cdot \frac{\partial}{\partial t_3}. \quad (20.14)$$

The following conditions are equivalent:

- $\alpha)$ F is holomorphic on C and E is holomorphic on M ,
- $\beta)$ $p_3 = 2$ or $(p_2 = p_3 \geq 3 \text{ and } g_i = 0 \text{ for } 1 \leq i < p_3 - 2)$,
- $\gamma)$ the curve $(C', 0)$ is weighted homogeneous.

b) Each irreducible germ (M, p) of a massive F -manifold such that $T_p M \cong Q^{(1)}$ and such that (L, λ) has 3 components is isomorphic to a finite number of normal forms as in a). The numbers p_2 and p_3 are determined by (L, λ) . The number of isomorphic normal forms is $\leq 2p_2$ if $p_2 > p_3$ and $\leq 6p_2$ if $p_2 = p_3$.

Proof: a) i) Direct calculation.

ii) \mathbf{a}_C is an isomorphism because b_2 is a unit in $\mathbb{C}\{t_2, t_3\}$. One can apply Corollary 15.2 because of (20.11). For the analytic spectrum see Theorem 20.3 a).

iii) The branched covering $(C, 0) \rightarrow (M, 0)$ is branched along $\{(x, t) \mid x = t_2 = 0\}$. Compare Theorem 20.3 c). The generating function $F : C \rightarrow \mathbb{C}$ has three different values on $\pi_C^{-1}(t)$ for $t \in M$ with $t_2 \neq 0$ because of (20.11) and the definition of f_i . Therefore $\mathcal{K} = \mathcal{B}$.

iv) (20.14) can be checked by calculation. $\alpha) \iff \beta)$ follows. Corollary 7.5 b) shows $\alpha) \iff \gamma)$ (one can see $\beta) \iff \gamma)$ also directly).

b) We start with coordinates (t_1, t_2, t_3) for (M, p) as in the proof of Theorem 20.3 a). The proofs of Theorem 20.3 a) and Lemma 15.3 b) give a unique construction of data $(Z, 0) = (\mathbb{C}, 0) \times (\mathbb{C}^3, 0)$, $(C, 0) \subset (Z, 0)$ and

$$\alpha_Z = dt_1 + xdt_2 + (b_2(t_2, t_3)x^2 + b_1(t_2, t_3)x + b_0(t_2, t_3))dt_3 \quad (20.15)$$

as in Corollary 15.2 for the germ $(M, p) \cong (\mathbb{C}^3, 0)$ of an F-manifold.

$(C, 0) = \bigcup_{i=1}^3 (C_i, 0)$ is the union of 3 smooth varieties, which project isomorphically to $(\mathbb{C}^3, 0)$, and is isomorphic to the product of $(\mathbb{C}^2, 0)$ and $(C, 0) \cap \{(x, t) \mid t_1 = t_3 = 0\}, 0)$.

The components $(C_i, 0)$ can be numbered such that the intersection numbers of the curves $(C_i, 0) \cap (\{(x, t) \mid t_1 = t_3 = 0\}, 0)$ are $p_2 - 1$ for $i = 1, 2$ and $p_3 - 1$ for $i = 1, 3$ and for $i = 2, 3$. The numbers p_2 and p_3 are defined hereby and satisfy $p_2 \geq p_3 \geq 2$.

The 1-form α_Z is exact on C_{reg} and can be integrated to a continuous function $F : (C, 0) \rightarrow (\mathbb{C}, 0)$ with $F|_{C_i} = t_1 + f_i$ for a unique function $f_i \in \mathbb{C}\{t_2, t_3\}$. Then $C_i = \{(x, t) \mid x = \frac{\partial f_i}{\partial t_2}\}$ and

$$\frac{\partial f_i}{\partial t_3} = b_2 \cdot \left(\frac{\partial f_i}{\partial t_2}\right)^2 + b_1 \cdot \frac{\partial f_i}{\partial t_2} + b_0. \quad (20.16)$$

We will refine (t_1, t_2, t_3) in several steps and change $Z, C, \alpha_Z, f_1, f_2, f_3$ accordingly, without explicit mentioning.

1st step: (t_1, t_2, t_3) can be chosen such that $(C, 0) \rightarrow (\mathbb{C}^3, 0)$ is branched precisely over $\{t \in \mathbb{C}^3 \mid t_2 = 0\}$.

2nd step: t_1 can be changed such that $f_1 = 0$.

Then $C_1 = \{(x, t) \mid x = 0\}$ and

$$C_1 \cap C_i = \{(x, t) \mid x = 0 = \frac{\partial f_i}{\partial t_2}\} = \{(x, t) \mid x = 0, t_2 = 0\}. \quad (20.17)$$

Because of $f_i|_{C_1 \cap C_i} = f_1|_{C_1 \cap C_i} = 0$, the functions f_2 and f_3 can be written uniquely as $f_2 = t_2^{\tilde{p}_2} \cdot \tilde{g}$, $f_3 = t_2^{\tilde{p}_3} \cdot g$ with $\tilde{p}_2, \tilde{p}_3 \geq 1$, $\tilde{g}, g \in \mathbb{C}\{t_2, t_3\} - t_2 \cdot \mathbb{C}\{t_2, t_3\}$. Now (20.17) shows $\frac{\partial f_i}{\partial t_2} = t_2^{\tilde{p}_i-1} \cdot \text{unit}$ and $\tilde{p}_i \geq 2$. Therefore $\tilde{p}_i = p_i$ and \tilde{g} and g are units, with $g(0) \neq \tilde{g}(0)$ if $p_2 = p_3$.

3rd step: t_2 can be chosen such that $f_2 = t_2^{p_2}$.
 (20.16) yields $b_0 = 0$ for $i = 1$ and $b_1 = -b_2 \cdot p_2 \cdot t_2^{p_2-1}$ for $i = 2$ and

$$t_2^{p_3} \frac{\partial g}{\partial t_3} = b_2 \cdot t_2^{p_3-1} (p_3 g + t_2 \frac{\partial g}{\partial t_2}) t_2^{p_3-1} (p_3 g + t_2 \frac{\partial g}{\partial t_2} - p_2 \cdot t_2^{p_2-p_3}) \quad (20.18)$$

for $i = 3$. The first, third and fifth factor on the right are units, therefore we have $\frac{\partial g}{\partial t_3} = t_2^{p_3-2} \cdot \text{unit}$.

4th step: t_3 can be changed such that $g = \sum_{i=0}^{p_3-2} g_i \cdot t_2^i + t_2^{p_3-2} \cdot t_3$.

We have brought the germ (M, p) to a normal form as in a). The numbering of C_1, C_2, C_3 was unique up to permutation of C_1 and C_2 if $p_2 > p_3$ and arbitrary if $p_2 = p_3$. The choice of t_2 was unique up to a unit root of order p_2 . Everything else was unique. \square

Remark 20.5. Results of Givental motivate some expectations on the moduli of germs of F-manifolds, which are satisfied in the case of Theorem 20.4.

An irreducible germ (M, p) of a massive F-manifold is determined by its restricted Lagrange map $(L^{(r)}, \lambda^{(r)}) \hookrightarrow (T^*M^{(r)}, T_p^*M^{(r)}) \rightarrow (M^{(r)}, p^{(r)})$ (chapter 9). Suppose that (M, p) is 3-dimensional with $T_p M \cong Q^{(1)}$. Then $(L^{(r)}, \lambda^{(r)})$ decomposes into a product of two Lagrange curves, a smooth one and a plane curve $(C', 0)$ (Theorem 20.3 a), Proposition 7.4).

If we fix only the topological type of the curve $(C', 0)$, we can divide the moduli for the possible germs (M, p) into three pieces:

- i) moduli for the complex structure of the germ $(C', 0)$,
- ii) moduli for the Lagrange structure of $(C', 0)$,
- iii) moduli for the Lagrange fibration in the restricted Lagrange map.

Within the μ -constant stratum $S_\mu = \{q \in M \mid T_q M \cong Q^{(1)}\}$ of a representative M , the modules of type i) and ii) are not visible because the Lagrange structure of the curve $(C', 0)$ is constant along S_μ .

But the modules for the Lagrange fibration are precisely reflected by S_μ because of a result of Givental [Gi2] (proof of Theorem 3): as a miniversal Lagrange map, the restricted Lagrange map is stable with respect to small changes of the Lagrange fibration which preserve the symplectic structure; that means, the germ of the Lagrange map after such a small change is the restricted Lagrange map of (M, q) for a point $q \in S_\mu$ close to p .

In view of Theorem 14.2 and Theorem 20.3 b)+c) there is 1 module of type iii) if $\text{mult}(C', 0) = 3$ and no module of type iii) if $\text{mult}(C', 0) = 1$ or 2.

Fixing the complex structure of the plane curve $(C', 0)$, the choice of a Lagrange structure is equivalent to the choice of a volume form. Equivalence classes of it are locally parametrized by $H_{Giv}^1((C', 0))$ ([Gi2](Theorem 1), [Va]), so the number of moduli of type ii) is $\mu - \tau$ (Theorem 7.2 b)). It is equal to the number of moduli of right equivalence classes of function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ with $(f^{-1}(0), 0) \cong (C', 0)$.

The μ -constant stratum of a plane curve singularity in the semiuniversal unfolding is smooth by a result of Wahl ([Wah], cf. also [Mat]) and its dimension depends only on the topological type of the curve. So one may expect that the number of moduli of types i) and ii) together depends only on the topological type of $(C', 0)$ and is equal to this dimension.

(But a canonical relation between the choice of a Lagrange structure and the choice of a function germ for a plane curve $(C', 0)$ is not known.)

In the case of Theorem 20.4 these expectations are met: the topological type of the plane curve is given by the intersection numbers $p_2 - 1$ and $p_3 - 1$; the last one g_{p_3-2} of the complex modules is of type iii), it is the module for the μ -constant stratum and for the Lagrange fibration; the other $p_3 - 2$ modules (g_0, \dots, g_{p_3-3}) are of type i) and ii). One can check with [Mat](4.2.1) that $p_3 - 2$ is the dimension of the μ -constant stratum for such a plane curve singularity.

Finally, at least a few examples of germs (M, p) of F-manifolds with $T_0M \cong Q^{(2)}$ shall be presented.

Proposition 20.6. *Consider $M = \mathbb{C}^3$ with coordinates (t_1, t_2, t_3) and T^*M with fiber coordinates y_1, y_2, y_3 . Choose $p_2, p_3 \in \mathbb{N}_{\geq 2}$. Then the variety*

$$\begin{aligned} L = \{(y, t) \in T^*M \mid & y_1 = 1, y_2(y_2 - p_2 t_2^{p_2-1}) = y_2 y_3 \\ & = y_3(y_3 - p_3 t_3^{p_3-1}) = 0\} \end{aligned} \quad (20.19)$$

has three smooth components and is the analytic spectrum of the structure of a simple F-manifold on M with $T_0M \cong Q^{(2)}$. The field

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{p_2} t_2 \frac{\partial}{\partial t_2} + \frac{1}{p_3} t_3 \frac{\partial}{\partial t_3} \quad (20.20)$$

is an Euler field of weight 1.

Proof: One checks easily that $\alpha = y_1 dt_1 + y_2 dt_2 + y_3 dt_3$ is exact on the three components of L , that the map $\mathbf{a}_C : \mathcal{T}_M \rightarrow \pi_* \mathcal{O}_L$ is an isomorphism of \mathcal{O}_M -modules, and that E in (20.20) is an Euler field.

The weights of E are positive. This shows via the Lyashko-Looijenga map that (M, \circ, e) is a simple F-manifold (cf. the proof of Theorem 18.4 a)). \square

Remark 20.7. In [Gi2](Theorem 15) the restricted Lagrange maps of two other series of simple F-manifolds with $M \cong \mathbb{C}^n$ and T_0M not a Frobenius manifold are given, the series Ξ_n ($n \geq 3$) and Ω_n ($n \geq 4$) (also $\Xi_1 = A_1$, $\Xi_2 = H_2$, $\Omega_2 = A_2$, $\Omega_3 = H_3$).

They have Euler fields of weight 1 with positive weights. The analytic spectra of Ξ_n and Ω_n are isomorphic to $\mathbb{C} \times \Sigma_{n-1}(2n-1)$ and $\mathbb{C}^2 \times \Sigma_{n-2}(2n-3)$, respectively. Here $\Sigma_k(2k+1)$ is the *open swallowtail*, the subset of polynomials in the set of polynomials $\{z^{2k+1} + a_2z^{2k-1} + \dots + a_{2k+1} \mid a_2, \dots, a_{2k+1} \in \mathbb{C}\}$ which have a root of multiplicity $\geq k+1$ ([Gi2] p 3256). It has embedding dimension $2k$.

The germs $((M, 0), \circ, e)$ are irreducible for Ξ_n and Ω_n , the socle $\text{Ann}_{T_0M}(\mathbf{m})$ of T_0M is the maximal ideal $\mathbf{m} \subset T_0M$ itself in the case of Ξ_n and has dimension $n-2$ in the case of Ω_n .

Givental [Gi2](Theorem 15) proved that the germs $(M, 0)$ for Ξ_n and Ω_n are the only irreducible germs of simple F-manifolds whose analytic spectra are products of smooth germs and open swallowtails. Generating functions in the sense of Definition 15.4 are due to O.P. Shcherbak and are given in [Gi2](Proposition 12).

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